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Encyclopaedia of the Philosophical Sciences

EDITED BY
WILHELM WINDELBAND AND ARNOLD RUGE

ENGLISH EDITION UNDER THE EDITORSHIP OF
SIR HENRY JONES

VOLUME I

LOGIC

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BY

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PREFACE.

I WISH to indicate in a few words the main purpose which the *Encyclopaedia of the Philosophical Sciences* is intended to serve, and the special features which, owing to that purpose, distinguish it from other works of the same class.

In the first place, each volume of the Encyclopaedia will deal with one main philosophical subject, including amongst others *Logic*, *Ethics*, *Aesthetics*, *The Philosophy of History* and *The Philosophy of Religion.*

In the second place, each volume will consist, not of brief articles, summary in character, dealing with a great variety of topics many of them of secondary importance, and intended to convey philosophical "information," but of original and relatively exhaustive discussions of fundamental aspects of each main subject. The Articles, as in the present volume, will be few in number. But they will be written by some of the most eminent philosophical thinkers in Europe and America; and, above all, they will be the exposition of the determining principles of their thought upon the great departments of philosophical speculation, and at the same time indicate their attitude towards the living experience of their times.

Nowhere is there greater diversity of view than amongst philosophers; nevertheless, there is no such witness to the unity of human interests as philosophy itself. In fact it is the mission of philosophy to realize this unity in the sphere of reflexion.

At no time in the history of the human race has either the need or the opportunity of philosophy to fulfil its mission been so great as at present. On the one hand, the surface tendencies of the day seem to be altogether towards divergency. Whether we survey "the world of thought" or that of "practice," we discover the same process of greater and greater specialization, and of segregation. The field of the thinker's inquiry is

TRANSLATOR'S PREFACE.

In endeavouring to present to the English readers of this volume a version which, while dealing faithfully with technicalities, should present the complex thoughts of the various writers in a form at once accurate and readable, I found myself confronted with difficulties which would at times have been overwhelming but for the unfailing help and encouragement of the English Editor. This was notably the case in the articles by Dr. Ruge and Prof. Windelband when language seemed at times almost incapable of expressing what they wished to communicate. I am glad to have this opportunity of offering my very warm thanks to Sir Henry Jones, who most kindly read through all the proof-sheets, and from whom I received throughout innumerable suggestions and solutions of difficulties. I may add, for the solace of the reader, that only two of the articles contained in the present volume, *i.e.* those by MM. Croce and Enriques, have suffered a double process of translation. M. Couturat's article was translated from his own MS., while that by M. Losskij was written in the first instance in German. I hope that in passing through my hands these articles, expressing as they do so many differing points of view, may have retained as much of their native flavour as is compatible with the process they have undergone, and that they will awake in their English readers no less interest than they have done in their translator.

B. ETHEL MEYER.

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INTRODUCTION

BY

ARNOLD RUGE.

THE arrangement and composition of an Encyclopaedia of the Philosophical Sciences demands from its Editor more than a systematic justification of his undertaking. He must also, and above all, give the grounds of the implications contained in the very name and title. It would seem certain that, before any science can hope to attain, or lay stress upon, a fairly secure starting-point for further progress by means of a Bibliography or Encyclopaedia, it must have already reached a certain definite stage of development. Even in the drawing up of a Bibliography, which involves nothing more than an apparently mechanical and technical collecting and arranging of contributions already made, the conception of the science concerned must be firmly grasped; for it is not a question of putting together everything which has been produced on the subject, but of selecting that which falls within a definite and comprehensive conception of the science. Only in that way can the historical progress of the particular science be secured and the contributions already made serve as a criterion for later writers. If, then, the modest compilation of a Bibliography involves a certain stage of maturity, of unity of direction on the part of a science both as regards matter and method, how much more the compilation of an Encyclopaedia! But if we may confront the task of collecting together the results of a particular sphere of scientific research not without joy and confidence, we cannot attempt to gather together and present as final the results of a period of philosophical speculation without much misgiving. For, however we may understand the idea of Philosophy, whether as the comprehending of spirit

by itself or as the comprehending of something alien to itself, there always lurks in the notion of Philosophy the idea of an unceasing striving towards a unity which in its totality and timelessness can never be grasped and reduced to a fixed formula by finite minds. But if this idea of unity and timelessness be involved in the very notion of Philosophy, we can never hope to colligate results. We can only point to the multiplicity of points of departure and of paths which lead towards this unity. Hence the only object of our Encyclopaedia of the Philosophical Sciences is to give some idea of this vital striving towards the idea of unity, and by no means to record fixed results. We can speak here not of bringing together but of working together. *The Encyclopaedia of the Philosophical Sciences* shows us philosophical thinkers in their efforts towards unity, working for the idea of unity. It takes us into the innermost workshop of thought and leaves it to the individual to observe the lines of direction followed by these independent thinkers.

The necessity of thus formulating the task proper to an Encyclopaedia of the Philosophical Sciences will be apparent to everyone who is able to form any mental picture of the present conditions of philosophical thought. For more than ever before does philosophical thinking, under the pressure of the age, tend to split up and to descend to pure specialization, following in the wake of the special sciences, of the advance in technic, and of the general conditions of life; and we are threatened with a time when we shall no longer be able to follow the threads of this multifarious development.

On the one hand Philosophy borrows largely from the special sciences, both as regards method and content, and on the other the sundered and exclusive ways of life demand from Philosophy a view of the world as a whole which shall be of practical value, a strong crutch upon which to lean in its course through time. And these causes tend either to divert Philosophy from itself to the pursuit of ends which are special, or else to isolate it in a bleak loneliness, cut off from sources of nourishment and a stranger to the world. Should, however, our Encyclopaedia succeed in bringing into co-operation the acknowledged leaders of the age, we shall at any rate be able to recognize the main lines, perhaps even the central meaning of this development. To see clearly how these main tendencies

are developing ought to help us to overcome any one-sidedness which might come from the compulsion of special problems.

But if the idea of an Encyclopaedia is also to find justification outside its own sphere, amongst those who are not philosophers, and in its relation to the special sciences and unphilosophical life, yet the postulate of an inner ground, of a justification from within outwards, from Philosophy itself, must not be in the least degree weakened or modified. Hegel's *Encyclopaedia of the Philosophical Sciences* should above all force us to ask whether the conception which he so clearly defined has been made good. Is it fitting that we should, as it were, bring to a new birth the great thoughts developed by Hegel in his own particular manner; and carry them over into an age which is a hundred years older and—as it believes—a hundred years riper? This question as to an inner justification, laid upon us by the memory of that which was done by one of the greatest of thinkers, involves that of the justification of the *form* which the new Encyclopaedia must take.

It was easy for Hegel to see in his *Encyclopaedia* authentic evidence on the one hand of the unity of Philosophy and on the other of the assured position of philosophical speculation. As, on the Hegelian method, every system of Philosophy springs up naturally as the result of the preceding system, so every system of Philosophy is, for him, the whole of Philosophy, and the *Encyclopaedia of the Philosophical Sciences* is not only the ground-plan of the entire Hegelian philosophy but also of Philosophy in general. Hence, with Hegel, the unity of Philosophy within his *Encyclopaedia of the Philosophical Sciences* is secured, not only *a parte subjecti* (*i.e.* by the philosopher) but also *a parte objecti* (*i.e.* by the content of Philosophy). In the form of the new *Encyclopaedia of the Philosophical Sciences*, however, this thought of unity, *a parte subjecti*, is torn up and rejected, and many philosophers are substituted for one; but the idea of unity, *a parte objecti* (*i.e.* the unity of Philosophy itself towards which all the philosophers are striving) is held fast.

But in every fixed and closed philosophical system which claims to be a rational reflection both of Philosophy itself and of its bearing on concrete life, there lurks a tendency to one-sided appraisement and evaluation. It is precisely in Hegel that we meet this tendency in its rudest form. The Hegelian

Philosophy, appearing as it did at the dawn of an age which was to be characterized by the infinite differentiation of the special sciences, hypostasized logical thought once more, and made it omnipotent. It allowed to fade out of sight the consciousness, always present in all knowledge, that in the idea of knowing is implied that of not-knowing and of not-being-able-to-know, and in the will towards truth the affirmation of the untrue. The proud over-lordship of rational knowledge, which turned a blind eye to the limits of the irrational, has been destroyed by knowledge itself.

To revivify this instinct towards moderation is the aim of our new *Encyclopaedia.* It binds itself to, and allies itself with, that idea of the unity of Philosophy and of knowledge in general which more or less consciously informs all knowledge. It will strive to draw more distinctly the boundaries between knowledge in general and the person knowing. But in so doing it must emphasize the notion of the temporal development of truth and throw a clearer light on the connexion which exists and must exist between that which is timelessly and that which is temporally valid. For even though philosophical thought were to stretch itself to that last point of union which lies beyond all time, yet it has its birth in and rises out of the life of particular individuals in time. It is true that the value of philosophical thought rests in its timeless validity; nevertheless this value can only be attained in the process of overcoming the temporal. And this process of overcoming the temporal with regard to and in favour of the timeless we may watch in our own time developing in a manner peculiar to itself. The struggle of present-day philosophy with the rich results and the methods of the special sciences is more severe than the old combat of the individual against a dominant system, a commanding dogma, or, in the last resort, subjective moods. To extend these results and methods to their final consequences, to show how they are conditioned by subjective and temporal aims of knowledge and to rise beyond these to absolute values is a greater, less egoistic, but also a less eventful enterprise than the attempt to reduce one's own subjective world-conception to a general formula. This aim of the Philosophy of our own time will be reflected in a double manner in the new *Encyclopaedia of the Philosophical Sciences.*

In the first place, as already remarked, the thoughts presented

in the *Encyclopaedia* will be worked out not by any one thinker, but by a multiplicity of thinkers. And secondly, not every one of these thinkers will present a complete system of Philosophy; some will only sketch the ground-plan of one particular philosophical science. Moreover, as regards the selection of these workers, once again it is not any one individual or his advisers who have chosen them out of the multitude of philosophical thinkers; a much more powerful and more objective editor of the Encyclopaedia, namely, the present age itself, has made the choice. Any one who is to-day recognized as a philosopher may unfold the principles of his philosophizing in this *Encyclopaedia of the Philosophical Sciences.* Thus a time will come when this new *Encyclopaedia of the Philosophical Sciences,* after it has been completed and the principles of the individual disciplines have been fully developed within it, will exhibit the ground-plan of the Philosophy of the present age. For whoever is valued in an age works in that age: this is the definition of what is "temporally valid."

As touching the separation of the philosophic whole into particular spheres, we cannot do better than quote for our new *Encyclopaedia* the opinion expressed by *Hegel* in Par. 15 of his *Encyclopaedia*, and formulated as follows: "Every one of the parts of philosophy is a philosophic whole, a circle closing upon itself, but the philosophical Idea expresses itself therein as qualified or as a particular element. Hence, though the particular circle is in itself a Totality, it breaks the limits of its Element, and becomes the ground of a wider sphere. Hence the whole exhibits itself as a circle of circles, every one of which is a necessary moment of it, so that the system of their particular elements constitutes the whole Idea, which also appears in each of the particulars."

As every system of Philosophy must begin with the doctrine of thought, the *Encyclopaedia of the Philosophical Sciences* begins with the doctrine of thought-forms, namely, with Logic. Hence the first volume of the Encyclopaedia of the Philosophical Sciences has "Logic" for its sub-title, and treats of the principles of thought in general. How far it is possible, at the first onset, to realize the idea of the whole Encyclopaedia is a question for the critic rather than for the expositor of the subject.

THE PRINCIPLES OF LOGIC

BY

WILHELM WINDELBAND.

To discuss the principles of Logic within a strictly limited space is no small undertaking. For Logic is no exception to the other sciences. Its principles first gain their significance and value from the way in which they verify themselves in the process of establishing, ordering and setting forth their own special concrete system of doctrine. Since they themselves are from their very nature underivable, the evidence in their favour is only to be found as they prove applicable to the particular and manifold, which by their means are to be simplified and universalized.

A special inquiry into principles is, however, comparatively easy and free from danger in the case of a particular science whose main structure is relatively fixed and accepted. And we should perhaps have found ourselves in this position with regard to Logic about a century and a half ago. It then stood as a well-built edifice firmly based on the Aristotelian foundation, to which subsequent exposition had in the course of time contributed changes in the arrangements of its parts, or made more or less prominent additions.

But, as is well known, this state of things was entirely changed by Kant. The transcendental point of view which the Critical Philosophy introduced widened the logical problem, and this was only the first step in an entire change of principles which has been proceeding since that time in different and partly opposing directions. The position of Logic at the present day is the exact opposite of a uniform and commanding structure: its principles are fluid, the contradictions which are to be found between them involve not so

much individual dogmas as fundamental points of view and problems of method; and no view of these most difficult questions which a particular writer may present can hope to establish itself by means of a general discussion, unless it can verify itself by its pregnant formulation of the special material into which it is inquiring.

If, nevertheless, I have resolved—not without reluctance—to publish a critical survey of logical principles, the possibility of so doing is given in the attitude which I have been able to take up with regard to the different ways in which Logic has been treated. In the trouble and confusion of the present philosophical movement every one of these principles has, from its own standpoint, been developed in a more or less valid way. It would indeed be incomprehensible were there not at least some kernel of relevant truth in each of them. The error of one-sidedness only begins when that which is justifiable in its own place asserts itself as alone valid and as the complete truth,—when it thinks it ought to exclude all else. Any one who has looked on long enough at the interaction of different points of view, or has himself been engaged therewith, must be finally convinced that an exhaustive solution of the great aggregate of logical problems can only grow up out of the union of all the different methods of treatment to which Logic has been subjected in virtue of the inner essential manifoldness of its nature. But no weak putting together or eclectic indecision can bring about this union. What is required is a systematic whole, in which the different special problems and the principles whereby they can be solved are organically developed from the fundamental problem in their articulated order.

In order to do this, however, it is necessary to conceive the task of Logic in the most comprehensive way. It must be regarded as the Philosophical Doctrine of Knowledge, as the theory of Theoretical Reason. Metaphysics and Natural Philosophy which, according to the ancient division of Philosophy, were included under Physics, fall, according to post-Kantian thought, within the province of the Critique of Knowledge and Theory of Science; and if we are to regard these as integral parts of Logic then it comprises the entire contents of theoretic Philosophy. And on this account Logic cannot be confined to the abstract questions raised from the different points of view from which its aspects have been grasped. On the other

hand, these individual standpoints find their particular justification and their historical possibility precisely in the fact that each one in its place is grounded in the systematic continuity of a philosophical Theory of Knowledge.

For according to the Critical Method, through which alone there is assigned to Philosophy a problem and province of inquiry of its own, clearly marked off against all other sciences, philosophical thought is everywhere directed towards the task of inquiring into those activities of human reason by means of which, in the course of history, the entire structure of civilization has grown up. The object of such an inquiry is to discover how far general postulates of reason, which are independent of the specific conditions of humanity and which find their justification entirely in themselves, have attained to consciousness and effective value. Hence there are only three fundamental philosophical sciences: Logic, Ethics and Æsthetics, corresponding to the fundamental psychical activities of knowing, willing and feeling and to the forms which human culture has taken: Science, Morality and Art. For the psychical functions relevant to these provinces and their external manifestation in the life of mankind are all empirically given; and the critical reflection of Philosophy, starting from the foundation thus furnished in experience, must decide whether and how far the value of the contributions of human reason to this natural and historical structure has been over-estimated.

It follows that in indicating the main stages of the way from the *a posteriori* to the *a priori* for Logic, or the theoretical part of Philosophy, we must, if our assumption is valid, show how the different points of view from which Logic can be and has been treated are the necessary stages of this advance. And at the same time we must be able to show clearly, from the systematic inter-connexion of the whole, not only the right but also the limitations of the right of each one of these individual standpoints.

I. PHENOMENOLOGY OF KNOWLEDGE.

The empirical material for a philosophical discipline presents itself formally in two ways; on the one hand as a mass of immediate experiences of the pre-scientific consciousness, on the other as ordered systems of concepts which have been

already developed out of the former by the empirical sciences. Thus Ethics deals with the volitional experiences which are familiar to every individual—with his moral judgments and his general moral and legal relations. But it also deals with what it learns from Psychology as to the action of motives, from Jurisprudence as to the historical and systematic establishment of legal institutions, from Ethnology and the science of History as to the development of morality and all the changing forms of the relation between the individual and the social will.

The empirical data of Philosophy have a two-fold character in another respect. This necessarily arises from the fact that the rational activities of the human mind appear on the one hand as functions of the individual consciousness which are everywhere the same and which are governed by natural laws, on the other as results of the entire historical life of the race. Thus the data of philosophical Æsthetics consist, on the one hand, of the processes of intuiting and feeling, of enjoying and creating; on the other of the forms taken by Art among different nations and of the historical inter-connexion of its origin and its valuation.

The distinction between the data of experience and the data of empirical theory, and also the distinction between what is given in the mental nature common to all men and in the historically differentiated forms of humanity reappear, although of course with a certain fluidity of outline, within the province of the Phenomenology of Knowledge. We understand by this latter term the sum total of the empirical phenomena of knowledge which constitute the given presuppositions of Logic as theoretical philosophy. We find these first of all in the familiar processes of the individual consciousness which we all mean when we speak from our immediate experience of knowing; we find them also in the theories which the empirical sciences, led by Psychology, have developed in order to describe and causally explain these phenomena. But, in addition to these, the facts which underlie Logic are given in the entire group of sciences; for the sciences exhibit the historical forms of human knowledge, and it is in or by them, in their historical sequence that logical thought has progressively sought to determine the nature, the meaning and the value of knowledge and of science.

If, now, we survey critically the different stages of the Phenomenology of Knowledge we must start from the *fundamental fact* which underlies all logical reflection. It is that we make a *distinction, from the point of view of value, between the true and the false.* But though this fundamental presupposition of Logic may meet with universal acceptance in this very general form, yet we shall soon see that, in its different moments, it requires a more exact definition. For, on the one hand, it may be asked what precisely are the ideational forms to which this logical distinction of value between true and false is applicable; and on the other, though we may in practice be tolerably agreed as to what we mean when we attribute or deny the value of truth to certain mental forms, yet directly we try to form a more exact conception, and to formulate more definitely the meaning of this evaluation we find ourselves involved in difficulties whose solution can be expected to follow only from the very latest and most delicate developments of the problems of Logic. So we see at once that even these phenomenological prolegomena cannot be treated without reference to ultimate questions, and that all those attempts which restrict themselves to one of these empirical stages are necessarily inadequate.

This is pre-eminently true of the *psychological* treatment of logical problems. Psychology must, in any case, supply the first foundation. For there is no doubt that it is as psychical processes that we are first aware of apprehending and knowing, and although Philosophy may apply to them her own particular method of treatment, yet she is compelled to take for granted that there are settled and exact terms for these universally familiar experiences; and this assumption is the more indispensable in proportion as the expressions for the different kinds and phases of mental activity are vague and indefinite in all languages. This condition of popular speech is indeed quite comprehensible and even unavoidable in face of the fineness and delicacy with which the manifold threads of mental life are graded and interwoven: hence the first demand which Logic (and by analogy, Ethics and Æsthetics) has to make on the preparatory labours of Psychology is the creation of a settled and unambiguous *terminology*, and this requirement is, precisely in the case of Logic, not yet perfectly satisfied. When, for example, it was said above that there are ideational forms the truth-value of which Logic has to discuss, and the question arises as to which

these forms are, the word idea (*Vorstellung*) is taken in the general sense (after Kant and Herbart), according to which it means the entire theoretic and disinterested functioning of consciousness, in contradistinction to those interested states of mind which appear as feelings or as acts of will. But this wide meaning of idea (*Vorstellung*) is far from being generally accepted: many psychologists and logicians oppose idea, taken in the narrower sense as "immediate perception," to thinking itself—as, for instance, it is said something may be conceived but not intuited (*vorgestellt*). Such regrettable disagreements, which necessarily involve logical uncertainties, are doubtless due to the fact that Psychology was for so long pursued by philosophers who laid the chief emphasis on its general problems and doctrines. Not till Psychology has become an empirical discipline, completely independent of Philosophy, so that different thinkers can work at it continuously, not till then may we hope that Logic (and Philosophy in general) will be able to speak of the psychological data which it has to accept from empirical science with the same exactitude and unambiguity as it can speak to-day, when working with the concepts of Mathematics and Physics. Until this goal is reached every logician, in order to safeguard his own inquiry and ensure absence of ambiguity, must begin by defining as clearly as possible the fundamental psychological concepts which he requires.

The next consideration, from the logical point of view, is a systematic terminology, and this is the business of descriptive Psychology. But such a formal arrangement as is herein involved can, for methodological reasons (see below, p. 51 f.), only be acquired and established, if it is to be scientifically satisfactory, by the genetic method. Hence Logic cannot be indifferent to *theoretical Psychology*, which has always occupied itself in examining how and by what stages judging and knowing, as the highest and most significant activities, have developed from the elementary beginnings of sensuous presentation. The most important presupposition of this psychogenetical inquiry is the view which, since the time of Locke, has been accepted almost as self-evident, and has seldom been called in question, viz. that the ultimate constituents into which we can analyse the always complex content of our conscious experience existed originally as simple elements. From this point of

view it is usual to take sense-perceptions as the psycho-physical foundation of all perceptual life, and the theory based on this goes on to discuss what are the gradated series and the laws according to which complex ideas are constructed out of these simple ideas, and, finally, how the abstract is derived from the concrete. This construction of the theoretical consciousness has been worked out most delicately and carefully in its various stages by the *Ideology* of the eighteenth century, in which the impulse to system-making which had shipwrecked on Metaphysics indemnified itself in various ways. The controversy of that time centred round the principle as to whether such a transformation of the lower forms into higher, of the elementary into finer states of consciousness, takes place automatically according to the laws of physical mechanism or of psychical chemistry (as the Associationist psychologists say), or whether the various faculties and finally consciousness itself as essentially continuous and one must also be invoked. The prolonged dispute as to innate ideas reduced itself to this question in the end. The decision of all these genetical problems, however, while for the Ideologist a matter of life and death, and for Theoretical Psychology of the greatest significance, is for Logic itself quite irrelevant; for Logic is concerned not with the origin but with the validity or truth of ideas. Logic is interested in these psychogenetic inquiries only in so far as they are necessary or fitted to make the different types of presentation-processes clear and distinct through their inter-relations. But if this evolutionary history of knowing, as it has actually taken place, is, as the Ideologists wished, and to a certain extent still wish, put in the place of Logic itself it only shows that they have not yet penetrated as far as logical problems. There are logical principles of Psychology (as of every science), but there are no psychological principles of Logic.

The most important point for Logic in the phenomenological survey of psychological pre-suppositions will always be the question as to the nature of the ideas with the validity of which it is concerned and what this validity itself signifies. These two questions, as any one may easily satisfy himself, are intimately connected; they cannot therefore be decided at this stage, but can only be discussed with a view to a preliminary orientation. The *naïve* consciousness is indeed only too ready to declare

presentations and also ideas, whatever their nature and origin, to be true in the sense that the presented content is real independently of the ideational activity, or that that which possesses an *esse in intellectu* must also be supposed to possess an *esse in re*. This view is then applied to concepts and judgments in the same sense as to the sense-perceptions out of which this connected view of the *naïve* presentation of the world has been developed. This conception of truth, which is usually defined as the correspondence of the idea with reality, I will call *transcendental* and the underlying theory the representative theory; its meaning is that the task of apprehending is to present the world as it is and this demand can of course be applied to any idea whatsoever, simple or complex, primitive or artificial. Nevertheless, a little reflection suffices to show to us that the employment of this conception of truth at once involves us in difficulties, for when we wish to test the correspondence we find we can only compare an idea with other ideas, never with its putative object. Moreover, there are actual truths, such as arithmetical propositions, for instance, of which not even the most ingenious explanation can show in what sense their content can be said to correspond with any kind of reality. So we get, side by side with the first concept of truth a second—the *immanent*; and it is self-evident that this truth is concerned not with any one idea but with the relation between different ideas. How far the relation between the ideational content and the so-called object of the *naïve* view of the world still more or less clearly stands in the background cannot here be inquired into. At this point it is of far more significance for us that, amongst the different kinds of ideas, the one which itself concerns the relation between ideas, namely the judgment, appears in the foreground of the logical field of interest.

But the psychology of the judgment had discovered even in antiquity, in addition to the act of thought which it undoubtedly contains, a still further moment which the Stoics called συγκατάθεσις: it is the affirmation or negation, the acceptance or rejection, the adoption or refusal of the content of the judgment. After some neglect this moment was again brought forward by Descartes, but it has only obtained full and express recognition amongst the logicians and psychologists of the present day; and even to-day we may still say that, as regards

terminology and in many respects also in essentials, no unanimous and unambiguous decision has been reached. If the judgment is regarded as a psychical activity there can be no doubt that both moments, the theoretical and the practical, as they have been called, are equally essential,—a conceived content and the attitude taken up towards its truth-value. In the psychological sense, however, the second moment is so essential that it constitutes the specific difference between the judgment and the remaining kinds of ideas, or of thought. Logical thought, on the other hand, which is concerned with the essential value of the conceptual content, and not with whether it is or is not recognized by empirical subjects, will remain inclined to determine the judgment as did Aristotle, essentially by its theoretical significance, and hence to treat its acceptance or rejection as an empirical and secondary determination. The main difficulty, however, lies in the fact that even Pure (or Normative) Logic, owing to that fundamental distinction between the true and the false, cannot separate affirmation and negation from the essence of the judgment. Hence it is that the attitude taken up by a logician on questions of principle is so largely determined by his attitude towards the quality of the judgment; hence, too, the explanation of the important *rôle* played by the theory of negation in modern logical literature.

The moment of "assent," however, presents still further aspects, which are extremely interesting for psychogenetic investigation. In its contrast to the theoretical act of thinking it exhibits itself partly as a function of feeling, partly as one of will, and as such it may, according to general psychological principles, be explained both as to character and origin in various ways. The ideational content has the characteristic of bringing with it the feeling of approval, and is hence regarded as *evidence*; while the feeling itself has several different shades of meaning, such as belief, the feeling of conviction, the feeling of validity, the *consciousness of validity*, etc. It is by means of this feeling that "true" ideas are distinguished from others and declared to be "valid." Acceptance and validity may here have as a secondary meaning either transcendental or immanent truth, but they may also be quite destitute of both these meanings, in which case they imply nothing more than immediate necessity of thought. We will call this last concept of truth *formal*, because

in itself it involves no sort of relation to objects. It is, however, a far-reaching and interesting task for Psychology to establish in which cases and according to what laws this feeling of value does actually occur. Opinion and belief as well as perception and knowledge fall within the sphere of this inquiry; indeed, the task of Psychology here is to establish the marks which from the psychological point of view distinguish the purely theoretical grounds on which perception and knowledge are accepted from those of opinion and belief. David Hume, in his *Treatise*, has carried through with exemplary subtlety the analysis of the ideational process in which belief is transferred from one idea to another by means of association: but precisely because of the purely psychological principle of his inquiry he was not able to establish any logically valid distinction between the different kinds of association.

Further, if the "acceptance" which is contained in the judgment be treated psychologically as a volitional act, the question as to its place in the teleological complex of conscious life, namely, as to its aims and motives, must be raised. The judging of ideas as true or false, implying as it does approbation or disapprobation, can only take place in a process of thought, which is either itself purposive or which pronounces upon what is not purposive, and it presupposes that truth exhibits some value for the judging consciousness. Now, psychogenetically, truth has no primary value for mankind. Like all the results of civilization it attains value by means of many media, and, in accordance with a general law, begins as means and becomes an end in itself. Undoubtedly it has value only within the kingdom of science and even there only for a small fragment of the body of investigators: for the great mass of men truth is still only a means to the attainment of all kinds of other aims. If we trace the stages by which truth attains value, whether in the individual or in the race, we see that it only becomes dear to man as he finds it useful and as he needs it to carry on his affairs; hence the direction and the sphere in which he seeks for truths have at all times been determined by his simple or more complex, lower or higher needs. We thus understand that for this psychogenetic and, in the last instance, biological examination all kinds of grounds for accepting an idea as true led to practical results, changing sensuous into motor processes in different ways. In this respect, therefore, perception

and knowledge had no advantage over opinion and belief. At this stage the only question is that of the strength and effectiveness of the feeling of validity, and we can understand why, both in ancient and in modern times, psychologists should substitute the use and the practical effects of ideas for their truth which, at any rate, theoretically, could not be clearly defined. It is in this niche in the entrance porch of Logic that Pragmatism with all its rhetoric has its home.

In these inquiries into the development of the truth-value we have already overstepped the sphere of individual conscious life and have reached the *socio-psychological* presuppositions of Logic. For perceiving and knowing as empirical functions are entirely social in their nature. They are integral parts of the common mental life—for the lonely strivings after truth of the individual are a late product of civilization which is always rooted in some historical community of knowledge and tends to discharge itself into it again. Hence among the marks of the concept of truth must be reckoned the universality of recognition or validity. This cannot indeed be actually universal, for it is hardly ever attained even within very narrow social connexions, and even were it attained it would be no guarantee of truth. But the claim to universal validity, which every affirmation of truth implies, and the reference to a plurality of subjects who judge are the outcome of the social community, and therefore also an empirical symbol of the actual necessity which the very notion of truth primarily implies.

The social character of cognition, however, shows itself above all in the fact that it finds its expression in *speech*, as the most characteristic vehicle of the common life. The connexions between thought and speech, therefore, form a most important subject for the Phenomenology of Knowledge, and what Psychology, Physiology and the Science of Language have to say about it cannot in its last results be a matter of indifference for Logic. All perception and knowledge are given us first of all as expressed in language; hence the question arises as to how far and in what way the inner processes, which are the essential ones for Logic, really attain to adequate presentment in that external form. For certainly, according to our experience, there is hardly an act of thought without at any rate a slight impulse towards speech, and we have no reason to assume that the processes of consciousness which go on in general images, still less in concepts and

generic concepts, are ever empirically given without the help of words. As a matter of fact every man learns to think as his powers of speech grow. But however dependent on speech thought may be as an actual function, yet it is neither entirely bound up with it nor essentially identical with it. Not only pathological states, such as aphasia, but also whole stretches of normal conscious movement both in imagination and in thought, when the idea or state of mind struggles in vain for expression in speech, prove that the content of consciousness is independent of speech. And on the other hand, speech may run on mechanically without bearing with it the corresponding movement of ideas in consciousness. In any case, however, speech is essentially different from thought. We have only to consider the multiplicity of languages to make this often overlooked relation clear. The didactic value of being able to speak several languages consists precisely in this, that the possibility of different expressions for the same ideational content is continually experienced; and with this the *naïve* belief, that the thing is necessarily given and self-evident in the word, falls to pieces. But it must, above all, be clearly laid down that the linguistic relational forms are nothing less than imitations of the forms of the movement and association of ideas. They are in themselves something quite different—namely, signs for these. And it is to this symbolical character that they owe their amazing mutability and capacity for modulation. Bound up with this is the fact that the relation between thought- and speech-forms is far from being always the same. On the contrary, as words are sometimes homonymous and sometimes synonymous, so also the same forms of speech sometimes stand for different forms of thought, and the same forms of thought are expressed in different forms of speech. Herein lies the marvellous secret of speech,—that its fluid indeterminateness, which is no small part of its aesthetic charm, is, as a general rule, in no way prejudicial to mutual understanding. Moreover, speech, whose business it is to give living expression to the spiritual community as a whole, and to all its interests, has many other purposes besides that of knowledge, and all of these have co-operated in forming it. At any rate the natural evolution of speech expresses the capricious play of ideas rather than the purposed processes of knowledge. It was the discipline of critical thought which first left traces of itself in the language of civilized people. In the light of all these

considerations we shall be able to decide what significance the phenomenology of knowledge has to attribute to the appearance in speech of thought and perception. The fundamental logical act, the judgment, finds its verbal form in the proposition; hence it is historically comprehensible why the first logical theories did not get beyond the analysis of the proposition, the discovery of its constituent parts, its forms and kinds. This was the origin of the attempts at logical investigation made by the Greek Sophists, and we find echoes of this in Aristotle, and still more among the Stoics: the amalgamation of Logic partly with Grammar and partly with Rhetoric was, as is well known, revived in principle in *Ramism*, and has also appeared later here and there. In opposition to this, as we have already seen, we may say that, though logical form dawned upon consciousness hand in hand with speech form, yet it must never be confused with the latter. The relation is, indeed, rather the reverse—the riper forms of linguistic culture, so far as we are concerned with the life of ideas, can really only be understood by means of their logical significance. There are certainly logical principles of Grammar, but there are no grammatical principles of Logic.

The psychical and linguistic forms of knowledge are to be found within the entire circle of the ideational life where and whenever it is a question of knowledge and perception, whether that knowledge be purposive or not. But from this great mass of material those historical forms emerge which, as *Sciences*, make up the narrower object of logical inquiry. For by Science we understand that knowledge which knows itself as such, being as conscious of its aim as of its grounds, of the problem it has to solve as of its manner of knowing. It is on these grounds that there falls within the sphere of logical values all that knowledge which has been brought into existence by the experience and reflexion of daily life; nevertheless, in its own proper field, Logic appears as the philosophical theory of science, and it is in this sense that the sciences—as they exist as historically evolved facts—form the empirical foundation by which Logic has to orientate itself. We cannot state too clearly at the outset that Logic must never dream of interfering with the work of the special sciences (which indeed it has very seldom attempted to do). It is in no wise its object to shake their foundations,

but, taking them as actual knowledge, to study their philosophical significance. How much this amounts to positively can only be really developed in Logic itself, partly as Methodology, partly as Theory of Knowledge. In this preliminary phenomenological discussion all we have to do is to repudiate *in toto* those unjustifiable claims on the other sciences which have been imputed to Logic. At the same time it equally behoves us to make quite clear that Logic does not content itself with merely registering the methods of procedure of the different sciences. Nor does it study the actual theories with which they occupy themselves in order to distil out of them general and abstract results. Every science, it is self-evident, must have its own ways and purposes. Occasions will not be wanting to any discipline when, for instance, it will feel the necessity either of considering, *à propos* of new problems as they arise, its method of treatment, or of finding a systematic form for an instructive synthesis of results arrived at, etc., etc. Thus within every special science, whether worked out by itself or not, lies a method, and hence a fragment of Logic; and modern Positivism is inclined, as Comte avowedly did, to regard the ascending series of these methods as an alternative for a special science of Logic. But to do this is to lose sight of the fact that in the selecting and mustering of the disciplines thus collated general points of view are assumed which would not be obtained from any one of these special sciences; and the recognition of this fact leads us back once more to the field of Logic as a special, that is to say, as a Philosophical, science.

This sketch of the Phenomenology of Knowledge was almost bound to lead us into polemics, because we had to show that within these spheres are to be found materials only, not principles of Logic. It follows from this that all those ways of treating Logic which do not advance beyond one or other or even several of these preliminaries do not do justice to the philosophical problems of our science. On the other hand, I should like, in conclusion, once more to emphasize the fact that all these phenomenological preliminaries—the fixing of a psychological terminology, the tracing of the genetic process which produces the feeling of value, the insight into the delicate relations existing between thoughts and their verbal expression, the comprehensive knowledge of the methods of work in the

different sciences—that all these preparatory labours are indispensable for any logical inquiry and theory.

But such a theory and its articulation must start from the most general character of the theoretic consciousness. We find this in Kant's *principle of synthesis*. Every idea, at every stage, has a multiplicity of moments, however limited, which, while distinct from one another, are yet connected with one another by some sort of relation into a unity. The content of consciousness is never simple and indivisible; it is the indivisible act of ideating which brings the plurality of the contents into a unity—the synthetic unity—higher than any which could be brought about by a formal connexion. On this rests the fundamental distinction between the *content* and the *form* of an idea. These must not be understood as two separate psychical realities, which unite with one another in the act of knowing: as, for instance, it has been held that the form was the permanent, the content the changing reality within consciousness. It would be truer to say that among psychical facts there can never be found a single form which is not that of a manifold of contents united together by it; or a single content which has not many elements, which are only bound together into a synthetic unity by a form. It is only abstract thought that can separate form from content: but since in doing so it turns the form itself into a content (or "object") of an idea, it cannot help thinking it again in turn under some form; while in trying to isolate the content and to separate it from its form, it is obliged to think the various moments in another relation, *i.e.* under another form. But in this abstraction we see once more how peculiarly complicated is the relation between form and content. Often enough the same form may embrace very different contents, and occasionally the same content may appear in different forms, and it is this which enables them to be separated in abstract thinking. On the other hand, it is not possible to apply every form to any multiplicity of moments of content, nor is every content compatible with any kind of form. And here we come on an inner and essential relation between form and content, the investigation of which (till now rather overlooked) lies on the borderland between Psychology and Logic: for this relation embraces the whole gamut of possibilities from the clear and necessary formulation of the content, to that which is contingent and

allowed by it, or even to that which it itself rejects. The closer inquiries which are demanded by this perhaps most difficult problem of the theoretical consciousness can, of course, here only be indicated in the most general way. In a certain sense we may say that the course of logical theory, as it will be sketched out in the following pages, consists precisely in the systematic advance from the analysis of the form of thought to the understanding of the relation which exists between this form and its content.

Our first task, therefore, is to isolate by abstraction those forms of thought which are indispensable to the attainment of truth in perceiving and knowing, and to exhibit their immediate evidence. We call this part of our inquiry, *Formal or Pure Logic*, inasmuch as here we have to abstract from every relation to any particular knowledge-content, although, of course, not from content in general, which would be impossible. The forms so discovered are valid for all kinds of thinking which have truth as their object, whether pre-scientific or scientific; and as we can still pay no attention to the especial subject-matter, we are dealing with the kind of truth which we have on that very ground decided above to call formal.

It is the second part of Logic, namely Methodology, that deals with the particular content of knowledge and the objects for which it stands; for the task of Methodology is to exhibit the purposive inter-connexion of logical forms, by which the particular sciences attain their end with regard both to the formal and the essential nature of these objects. And we must show in how many different ways the individual disciplines are able to exhibit the systematic inter-connexion of all the elements of knowledge which lie within their province. In this sense Methodology is chiefly concerned with immanent truth, *i.e.* the agreement of ideas amongst themselves.

Finally, out of the labour of the sciences grows up the world-conception which, in opposition to the subjective opinions and convictions of the individual, has a purely theoretic foundation. Philosophical criticism has not to question the objective validity of this conception, but rather to ask as a final problem how such an objective world-conception is related to the absolute reality which, according to the pre-suppositions of the *naïve* consciousness, forms its object. The *Theory of Knowledge* to which we assign this problem can bring forward for its solution

no other arguments than those which are offered by the special sciences themselves, and by the first two parts of Logic. Only by means of these shall we finally be able to decide whether and how far and in what sense human knowledge guarantees that transcendental truth which hovers in the background as the undefined presupposition of all pre-philosophical knowledge.

II. PURE OR FORMAL LOGIC.

Pure Logic, or Logic in the narrower sense of the word, is generally defined as the doctrine of the forms of thought. But this definition requires qualification, and we must make clear at starting that Logic deals only with the *forms of right thinking*, with those which are deliberately selected from all the psychologically possible forms of ideational movement, as conducive to the attainment of truth. It must not be supposed to teach how people actually think, but how they *should* think if they want to think rightly. Further, while this customary definition sufficiently marks out the difference in principle between Logic and Psychology, we must not overlook the reference implied in it to that fundamental part of empirical thought we have already touched upon, namely, that it is exposed to error, and has to decide whether the results it arrives at are correct or incorrect. And though this reference to the alternative between true and false cannot be excluded by any attitude we may assume towards logical problems, yet it is necessary to point out at the beginning that the validity of these forms must, in the last instance, be entirely independent of the strivings towards knowledge of the empirical and, more especially, of the human consciousness.

This agrees with the view which makes the *general validity* of the progress of thought the criterion between logical and psychological forms; but even this reference to the empirical plurality of subjects can have significance only as emanating from the inner and essential necessity of the logical. In any actual agreement about truth, the universality of thought, looked at fundamentally (as, for instance, by Socrates in contradistinction to the Sophists), only furnishes the empirical occasion and handle for the attempt to prove the truth. This is evident from the fact that universal validity is demanded rather than given: its value therefore is that of a derivative rather than an original

sign, and if we take it as our starting point it is only because we are obliged to start from the real life of ideas.

Finally, all these considerations determine the sense in which the logical thought-form may be described as the *norm* and formal Logic as a *normative* discipline. For, as a matter of fact, on the side which it turns towards empirical thought, Logic, as the art of right thinking, has to establish norms; but the significance and the basis, the original validity of these norms must be quite independent of whether there are subjects capable of erring, whose empirical ideas sometimes follow and sometimes conflict with them. And here we come on a double aspect of all logical laws: on the one hand they are rules for the empirical consciousness, according to which all thinking which has truth for its aim should be carried on; on the other they have their inner and independent significance and being, quite independent of the actual happening of ideational processes, which are or are not in accordance with them. We may call the latter their *value-in-themselves*, the former their *value-for-us*, where by "us" we understand not only us men but any individual subjects whatever who, like ourselves, have to distinguish among their ideas between true and false or correct and incorrect.[1] Starting from "us" the logical is an "ought," but this "ought" must be grounded in something whose value consists in itself, and which only through its relation to a consciousness capable of error becomes for this latter a precept, a norm.

A prominent example of this double aspect meets us on the threshold of Formal Logic as soon as we ask how we can at all count on coming to an agreement as to the forms of universally valid, that is, of correct thinking. For all investigation and reflexion, all proof or refutation both in general and with regard to the logical problem would be purposeless if the rational consciousness did not recognize a normative constraint as soon as an assertion is made. It is only through such a constraining norm that we can proceed from one assertion to others, "assertion" being used here to include affirmations and denials. I should

[1] E. Lask in his *Lehre vom Urteil* (Tubingen, 1912) which I unfortunately did not see until I was correcting my proofs, has made the happy suggestion that we should distinguish the two pairs of opposites, used promiscuously as a rule, in the sense of these two meanings (*a.a.O.* p. 13 ff.). The terminology employed by him seems to me on the whole more appropriate and to the purpose than the inverse of Bergmann (cf. *ibid.* p. 26), only I should prefer to use "false" in opposition to "true" (*Wahr*) rather than to "correct" (*Richtig*).

like to call this most general requirement of that thinking whose object is truth the *Principle of Consequence.* It includes a whole number of traditional formulae, for example, that with the ground the consequence is also given, that with the cancelling of the conclusion, the ground is cancelled, etc.—as subordinate specifications; but when stated in its general form, it expresses the fact that within the sphere of the logical march of thought the consciousness of validity, which in opinion and belief is produced by many and by many kinds of causes, must only be determined on *theoretical grounds.* But this primary demand of all thought that professes to be perception and hopes to become knowledge is but the side, normally turned towards empirical thinking, of the fact that there is in the thought-content itself such an immanent inter-connexion that if something is true then something else is true and something else not true. This inter-connexion of valid ideas which is determined by form alone is itself the ultimate reason why, when supporting or denying a thesis in argument, we can compel our opponent, on the ground of assertions already admitted, to acknowledge other assertions. This "objective" meaning (as I will call it for the sake of brevity, although extremely unwillingly, on account of the vagueness of the term) involves, as is easy to see, the fundamental problem of the Theory of Knowledge, while the subjective meaning offers at once a handle for Formal Logic, but also contains for the latter a presupposition which cannot be proved because it is itself the principle of all proof. The same holds good for the methodological treatment of all the rules of right thinking which Formal Logic can lay down; their proof lies partly in the evidence they furnish to the normal consciousness, and partly also in the logical consistency and consequence in which they stand to one another as a system. But if any man stumbles at the fact that when we want to think about thought we must, in so doing, already follow the norms of right thinking—there is no arguing with him.

The matter is less simple when we come to the exhibition and the methodical investigation of logical forms. Here Phenomenology must supply the guiding thread according to which reflexion on the norms follows the principle that for every phase of the actual movement of ideas conditions are brought into consciousness under which they are recognized as generally valid and, in the normative sense, necessary for thought. The

primitive connexion between Logic and Grammar has had in this respect a long-lasting effect: as in the synthetic reconstruction of speech words were taken as the elements, the sentence as their combination and the paragraph as the combination of sentences, so it was thought that Logic ought to start with the concept and proceed from the concept to the judgment and from the judgment to the conclusion. This trichotomy prevails to a great extent in the scholastic Logic even of the present day.

Here, then, we are met with the necessity of distinguishing methodologically between linguistic and logical forms, λόγος προφορικὸς and λόγος ἐνδιάθετος. The *concept* as a logical form, which must be carefully distinguished from the idea of the primitive consciousness expressed in a word, is always the result of a *judgment* on which it is founded. The concept thus attained, however, when it has received a fixed name, may afterwards be analysed, and it thus forms the ground of a judgment which attributes to it one of its own signs. Only these (in the Kantian sense) *analytical* judgments presuppose concepts as their ground, whereas the *synthetic* judgments, on the other hand, in which perception consists, form and found concepts. If we distinguish between the meanings of the linguistic expressions usually so promiscuously used, we may say: all perception (meaning that thinking which while it seeks the truth is still in a state of flux) terminates in synthetic judgments and thus produces concepts. These concepts embody the knowledge which has been secured once and for all; in further perception it may be made fluid once more and may be used in analytical judgments for new advances in thought. In this process it is not unconditionally necessary (for sometimes, indeed, it can only be secured artificially by awkwardness of speech) that what we have here called a concept should be expressed in a single word. Logically considered, the concept is perception congealed into knowledge: in the judgment and in perception we gain, in the concept and in knowledge we possess the truth. But for this reason the logical structure of the concept is identical with that of the judgment; only the different stages of the ideational life in its striving after truth generally find, as word and as proposition, different expressions. When I say: "the will is free" I think exactly the same relation between "will" and "free" as in the word freedom-of-the-will; but in the judgment I assert the truth of the relation while in the concept I may only think it

without taking up any attitude towards its truth-value. The concepts, however, which make up knowledge have been confirmed by means of perception, and hence they preserve the moment of validity. We must distinguish from these those auxiliary concepts which are formed and used provisionally in the course of investigation without any attempt to determine their value as truth: they remain for the time problematic or hypothetical.

If we add to these considerations the further one that *inference* is nothing else but a way of establishing judgments, and is indeed a judgment by means of judgments, it becomes clear from this point of view too that Formal Logic can be nothing else but a *doctrine of judgments.* But by judgment as the fundamental function of perception we must only understand that which Phenomenology has shown us (see above, p. 14), namely the evaluation of a relation between ideas, an act of the synthetic consciousness which is judged according to its truth-value. In a completed judgment the two moments are, as a matter of fact, always united. A plurality of ideational contents in relation to each other is also present when, as in certain verbal forms of fragmentary judgments such as, for example, impersonal or existential propositions, we appear to be dealing with only one idea as the object of the assertion. Such propositions, some of which seem to have no subject, some to have no predicate, have raised difficulties only because it has been customary, owing to the verbal form of the proposition, to regard the whole doctrine of the judgment as the attribution of a predicate to a subject.

This customary schematization, however, is not altogether harmless, simple and evident as it may seem. We have only to try to reduce any scientific exposition, sentence by sentence, to the formula, S is P or S is not P, and we shall soon see that living thought will not let itself be pressed into the schema. The great majority of the propositions that we think, speak and write present a much-articulated manifold of contents; these contents are related to one another in different ways and can only be transformed into the affirmation of a group of predicates of a group of subjects by an unnatural constraint. But even in those simple cases which Formal Logic treats as normal the affirmation is by no means univocal. Its significance is at best verbally indicated rather than expressed in the *copula.* For in reality the copula is only the substitute for the verbal

form in predicates like adjectives and substantives which do not admit of conjugation: here is one of the leading examples of those economies mentioned above (p. 18), where the same colourless form of speech serves for a number of very different forms of thought. The thought-form as such never comes to expression in speech and from the logical point of view the affirmation consists in attributing to the subject not the predicate but *the relation to the predicate.*[1] Of the errors which have arisen from the logically accidental use of the existential verb as copula this is not the place to treat at length: we will only mention here that verbally it is chiefly the apperceptive process that decides which of the two ideas is to be taken as the subject, drawing attention first to itself, and which as the predicate to be attributed to it. In reality the relation asserted between *a* and *b* can also be expressed as a relation between *b* and *a*, with the inversion of the relation, of course, where necessary. Hence, in the logical sense, all judgments are simply convertible, while in their verbal form this is not the case. It depends entirely on the kind of relation: when this is reciprocal, as *e.g.* in equality, nothing stands in the way of verbal conversion. If I say that $\sqrt{4}$ is equal to 2, I can just as well say 2 is equal to $\sqrt{4}$. If, on the other hand, I say of gold that it possesses the property of yellowness (which is the logical meaning of the proposition that affirms of the subject "gold" the predicate "yellow"), so I may equally well say of yellow it is a property of gold; but verbally the conversion of "gold is yellow" to "yellow is gold" would appear as incorrect, or at least not as the exchange of subject and predicate but only as an uncommon and inverse form of proposition. For the inherence which, in this case, is the logically expressed relation—the relation of the thing to its qualities, belongs to those forms of union in which the united contents are not interchangeable, and are not really of equal value. We may therefore speak of a natural and actual order of subject and predicate, which is independent of the apperceiving process. In the case of inherence this habit of thinking and speaking is indeed so strong that Aristotle could assert that a thing could never be the predicate of a proposition.

[1] On similar grounds Lask, in his *Lehre vom Urteil* (p. 58), has come to the conclusion that the logical predicate is in every case the category which is predicated of the whole "material of the judgment." In this way the Aristotelian and the Kantian significance of "category" would be approximated as nearly as possible.

If we are to escape the secret snares of speech we must define the judgment logically as the assertion of a relation;[1] and by so doing we should re-unite those two moments which our psychological analysis showed us to be the essential ones. But we arrive at the same result through a criticism of the division of judgments as they were taken over by Kant in his well-known table as the result of the dogmatic structure of Formal Logic. Since the investigations of Sigwart and Lotze, however, it can no longer pretend to be obviously true, as it did in the last century. It can easily be shown that the distinction of *quantity* does not concern the function of the judgment as such, but only a difference between subjects, whose value for knowledge concerns the doctrine of concepts and of inference (according to the customary theory), and, more especially, of methodology. In the case of *modality* the relations are rather more complicated. If *modality*, according to Kant, contributes nothing to the content of the judgment, but only concerns the value of the copula for thought in general, yet we must not forget to inquire how it differentiates itself from *quality*, which includes this determination of value in affirming or denying. As a matter of fact, in most discussions, quality and modality are very much confused. But where modality as a moment of the judgment must be assigned a significance of its own, there remains hardly anything else than the gradation of measure and kind in the grounds which the individual consciousness requires for its assertion—differences which, through verbal ambiguities (of "can" and "must"), are reflected in the sphere of relation. And so, finally, we are left with *quality* and *relation* as the two most pregnant points of view in the doctrine of judgments.

The doctrine of the *quality* of judgments leads us necessarily to the norms of affirmation and negation, which, under the name of *laws of thought*, are known as the most general logical principles. In this sphere, certainly, if we neglect certain difficulties arising from their verbal expression, it is easy to overlook their relation to anthropological qualities; nevertheless there remains the reference to some empirical consciousness or other, which is capable of erring, and on this account requires the form

[1] This would cut off at their roots all the unnecessary difficulties which have been started by the question (which has only arisen owing to a verbal confusion) whether and how far the copula should imply the existence of the subject and predicate.

of denial. But the last problem which arises for us out of this doubling of norms, to which reference has just been made, may be comprehensively expressed as follows: How can the determinations of objective validity, which in themselves are purely positive, become norms for the relations between affirming and denying?

For, first of all, we must admit that denial means something other than the mere rejection of an affirmative judgment. It is, indeed, true that the number of correct but aimless and senseless negative judgments can be increased at will to infinity, and that we only reasonably deny that which in some way or other is in danger of being erroneously affirmed. This point has rightly been emphasized in modern Logic; but it still remains an open question whether this proves that the act of denial is purely subjective in character and limited to a subject capable of error. The more I consider these relations the clearer it becomes to me that all the arguments which have been adduced are indeed valid for the occurrence and the actual process of denying in the empirical consciousness; but that, nevertheless, both in general and in every especial case of true denial there must exist some actual ground corresponding to it. The incompatibility which every negative judgment denotes, or, in other words, the failure of the attempt to bring the elements of the judgment together in thought, must somehow be implied in the elements themselves. And here that peculiar relation between the form and content of consciousness, according to which they have only a limited power of free movement over against one another (see above, p. 21), appears as a chief, and, as far as I can see, not further resolvable condition, which must be accepted as given for the logical evidence of assertion and denial in reality; and this means that there must be implied in negation a moment of essential validity which is independent of the movements of a possibly erring consciousness.

In normative Logic the relation between affirmation and denial is expressed in the *principle of contradiction*, which forbids the denial of what is affirmed and the affirmation of what is denied. It has indeed been thought that this veto is unnecessary because the affirmation and denial of the same content is as naturally excluded as, *e.g.*, desiring and detesting: on the other hand, it has been urged that we must not be forbidden to deny what we may have previously erroneously affirmed, and the con-

verse. Here again it is evident that *contradictory disjunction* must be in itself essentially valid in order to furnish a ground for the interdictions which result therefrom for the psychological movement of ideas. It is true of this latter that, while psychical motives for affirming may exist side by side with those for denying, the material ground for deciding can exist only in one of them: and though practically it may very seldom happen that any one affirms and denies the same content in one breath, yet the significant value of the Law of Contradiction appears when it is combined with the principle of inference, according to which nothing can follow from an assertion which is contradictory either to itself or to some other assigned assertion. These relations must be developed at length in the doctrine of proof and refutation. Nevertheless they justify even here the demand made and satisfied by Aristotle that the principle of contradiction also must be objectively formulated. But when this is given in either of the familiar formulae, "A is not not-A," or, "It is impossible that a thing can both be and not be," we get a metaphysical principle or an epistemological postulate by which it is meant that reality rejects a contradiction. It is not for Formal Logic to attempt to justify this axiom, which has a far wider bearing. For these reasons it is to be recommended that with the new terminology the law of contradiction should be expressed in the indifferent form: the assertion and denial of the same relation cannot both be true.

But we have so far only analysed half of the contradictory disjunction; its other half consists in stating that assertion and denial cannot both be false, *i.e.* that one or other must be true. This is expressed in the *Law of Excluded Middle*, the validity of which cannot be destroyed by any apparent, *i.e.* purely verbal exceptions. The only peculiar point here is that the Law of Excluded Middle is expressed as having objective validity only, and hence no norm can be deduced from it. From the standpoint of the empirical consciousness it is far more necessary that the possibility of affirming or denying unconditionally and without exception every possible thought-relation should be altogether rejected.[1] In the process of perception the case very often arises that neither affirmation nor negation can be justified, but both are forbidden by the logical conscience. For

[1] These distinctions are of special significance for the theory of the truth-value of disjunctive judgments.

it is at this point that the third so-called law of thought comes into force, *the Law of Sufficient Reason.* It expresses the logical demand that every assertion must have a universally valid ground, and hence is opposed to the multiplicity of psychical causes which produce in the individual the feeling that opinion and belief are true. But here again we must emphasize the fact that the universal validity of the ground does not mean any quantitative principle, but rather the actual necessity of thought. But with this, this law also takes on the character of a prohibition: we may not assert, that is, we may neither affirm nor deny where there is no sufficient ground, and this prohibition may, as with the ancient sceptics and with Descartes, be expressed as the law of suspended affirmation or of *problematical* relation.

Hence we can represent the relations imposed by the laws of thought upon the two moments of the judgment, as follows: a relation between two ideas can be regarded indifferently. Directly reference is made to its value as truth, we get the question arising in its verbal and thought-forms. The decision of the question is either the assertion in which it is affirmed or denied, or else the problematical relation in which (either provisionally or permanently) its insolubility is asserted. Theoretically the relation is thought in exactly the same way at all these five stages, and it may occur under the form of speech as a term, combination of words or proposition, in all cases with the same logical content. Which of the four kinds of attitude, however, is to be recognized as the "judgment" seems to me a question of terminology. Many thinkers have been inclined to regard the question as the judgment in a preliminary stage and therefore already as a kind of quality of judgment: others have not been willing to admit this on the ground that the decision belongs to the completed judgment, and the same argument has been brought forward against my proposal to rank the problematic relation as a third kind of quality by the side of affirmation and negation. If, in spite of this, I still maintain it, it is chiefly on account of the relation (discussed above) of this "critical indifference" to the principle of Sufficient Ground. That the latter is essentially the norm for the empirical, and therefore for what may be an inadequate form of consciousness, shows itself in this, that our real thinking must often violate it; since, where there is no sufficient ground for asserting or

denying, it has to content itself with *probability*, which may be defined as *assertion from insufficient grounds.* The theory oi this, however, belongs to Methodology, which has to show why science as well as life cannot rest content with the demand, which Logic makes, for suspension between two solutions oı the question.

Finally, we must point out that we encounter great difficulties if we try to give this principle of *ground* an objective form, although for the two other laws of thought it was comparatively easy to do so. For to transform this logical principle into the principle of *causality*, as was customary in the earlier Ontology, has now for so long been clearly recognized to be a destructive and dangerous error that we need not repeat the demonstration here. Perhaps we might invoke in this place the *Principle of Consequence.* But if we hoped by this means to find ourselves once more within the neutral sphere of validity we could speak, as we have already shown above, of a universal inter-connexion of ground and consequences as existing in itself and constituting the norm of affirmative thought. But it would be wrong to say that every valid idea has a ground in virtue of which it is valid. For to do so would be to conceive a ground which differs from the idea of which it professes to be the ground. And this is not open to us, for it would involve the familiar *regressus in infinitum.* On the contrary, if the principle of ground is to be carried through, we must always assume that that which is itself true contains within itself the sufficient ground of validity, and hence establishes the other as well. This relation exhibits itself for the empirical consciousness in the familiar distinction between *immediate and mediate certainty.* But even Aristotle knew, and it was one of the most valuable results he arrived at, that the two relations do not co-incide; but that, on the contrary, in the movement of actual perception there are adequate grounds which certainly do not spring from those which are valid in themselves.

The *relation* between judgments is the subject of the *Doctrine of Categories.* This forms the climax of all logical theory. Since Kant's time this has been the great central problem of which no generally accepted solution has yet been found. Ever since the guiding thread which Kant thought he had found in the old "table of judgments" has snapped, the

important point is, conversely, to find a principle from which the system of categories, and with this, that of judgments, can be deduced. But this seems to me, as I have already pointed out, in my *Festschrift* in Sigwart's honour (Tübingen, 1900) to be no other than that of *synthesis* which, as said above, constitutes the universal essence of consciousness, and indicates the ultimate condition under which connected thought is alone possible. Only by reflexion on this condition shall we succeed in discovering the highest forms of relation, the fundamental categories which can thus be articulated into special relations. The only possible principle of advance in the development of this system of categories is that the moments already won can be further related to and combined with one another. In this way the systematic development does not need to bring in determinations from without, and yet it can throw off at its different stages the forms of relation which are empirically well-known.

We may see, however, at the outset, that the whole system of relations must divide itself into two distinct series, between which a certain correspondence obtains. Indeed, if any one were to collect together everything that has been treated as relations or categories in the different logical doctrines, he would be forced to recognize the necessity for a real principle of division, such as Kant had in view in his four-fold division with its subordinate trichotomy of stages. Such a classification, where metaphysical or epistemological principles make their way into the categories of Logic, might easily be regarded as a division of the spheres of knowledge or of the spheres of objectivity. It was in this way that Plotinus set side by side with the Aristotelian categories those of the intelligible world, and that Hegel divided the self-development of the Idea into dialectical relations and the fundamental determinations of the content of the natural and spiritual world of experience. E. v. Hartmann has with great energy exhibited the parallelism of the categories throughout the three different spheres of the subjectively-ideal, the objectively-real and the metaphysical; while finally Lask (*The Logic of Philosophy* and the *Doctrine of Categories*), without entering upon the development of the different series, has brought forward a highly significant sketch of another trilogy of the categorical system, which he divides into the spheres of validity, of being, and of the super-existential (*Ueberseins*). How far a corresponding serial structure ought

to, or can be, carried out here it is not as yet possible to say with certainty.

This demand for a parallel structure seems to me to be met by the proposal to distinguish between *reflective and constitutive categories.* Its principle lies in the different relations of consciousness to its objects; that is to say, in the fundamental moment of perceiving, in which the sense of truth-value determines itself when any reference is made to an object. Hence we should call those categories constitutive or objective which are thought to be really existing relations between objects; and those reflective which, although determined by the special qualities of objects, exist at first as relations in consciousness and only for consciousness. In this sense the two kinds of categories may be distinguished as transcendent and immanent in their relation to truth; so that I would say that the constitutive categories are existential and the reflective are valid. It is the final task of the system of categories to reunite the two divided series and to discover the forms of thought in which the two fundamental categories, the valid and the existential, are combined into a unity.

From the point of view of the categories of reflexion, the first and the fundamental function of judgment is to distinguish; for to relate ideas in any other way we must distinguish them and keep them distinguished from each other. The habit in speech of expressing distinction in a negative proposition ought not to mislead us as to the logical sense of such propositions. The elementary and self-evident axiom, that every moment of consciousness must be distinguished from every other and maintained over against it in its idiosyncrasy has indeed been expressed as the Principle of Identity; but it seems better to reserve this term for the categories we shall find occupying the most important positions among the objective forms.[1] This presupposition, on which all other categorical thinking rests, acts as a norm which guarantees the identity of meaning in words expressing common ideas, and also gives fixity to individual ideas.

The limiting case of distinction is *equality.* In this case, as is evident, the contents declared to be equal must be distinguished in some way or another. From the many ways

[1] Cf. my treatise "Ueber Gleichheit und Identität" (on Similarity and Identity) in the *Sitz. Ber. der Heidelberger Akad. d. Wiss. philos. hist. Klasse,* 1910, Nr. 14.

and stages of distinguishing and identifying (usually called similarities and dissimilarities) the other categories of reflexion follow in two series, which may be called the *mathematical* and the *discursive*. In the former of these series the fundamental form for the synthesis of the manifold, which consists of distinct, equal moments, is *number*. From this are developed the further categories of *number*, or of *quantity*, with the fundamental relation of the *whole to its parts*, and the relations of magnitude with the different determinations of *measure* and of *degree*. How far in this and in subsequent extensions of the mathematical series the intuitable relations of time and space will come under consideration I will not here discuss; but one thing must be emphasized, that is, that within the logical series they never signify logical principles, but only spheres of application for logical principles, and, more especially, that the fundamental function of counting as a psychical act presupposes time in no other sense and in no other measure than any other synthesis of the manifold. The derivation of all these categories from the relations of distinguishing and identifying shows itself also in the fact that all mathematical judgments may be expressed as judgments of equality, and that this relation of equality (as that of distinction) is absolutely reciprocal. On this depends the exchangeability of the members of an equation, their capacity for being substituted one for the other, and with this the fundamental logical structure of all the doctrines of number.

In the discursive series there develops out of distinction and comparison the conceptual relations, and at this point in the theory of judgment the customary theory of the *concept* finds its proper place. For the first task of logical thinking is the transformation of experiences into concepts. This means analysis which distinguishes and synthesis which reconstructs. If in this way we become clearly conscious in the concept of what was previously effectuated in the intuition (so that we may here speak of a kind of ἀνάμνησις) we must bring out two chief points: first, that the unavoidable imperfection of the analysis necessitates a *selective spontaneity* in the synthesis, so that here already the "objects" which are further elaborated in thought exhibit themselves as products of the logical consciousness itself; and secondly, that these first concepts, while akin indeed in content and in form to the primary ideas

yet differ considerably from them, both in content which is limited by selection and also in form which is raised to a certain elaboration. Here we get at the root of the difficulties and misunderstandings which appear so frequently when the same word which originally stood for the primary idea is also used to denote the concept; here too we have the right, and to a certain extent indeed the duty, of science to assign names of its own to the concepts coined by itself.

In the concepts reciprocal action between distinguishing and comparing develops anew. Every concept is not merely a collection, but an ordered and related whole of elements, in which their intimate connexion (chiefly through a constitutive category) is thought, *i.e.* asserted. In this way they offer together the possibility of comparison, which, by *abstraction* from the different and reflexion on the similar marks, leads to the formation of generic concepts; these, again, must not be confused with the general ideas, often homonymous with them, of the uncontrolled ideational process. By continuing the process of abstraction, and on the other hand, by converting it into determination (for the theory of which Lotze's finely thought-out doctrine of the undetermined general marks first provided a satisfactory basis), there arises that well-known gradation of concepts which includes the relations of their subordination and co-ordination, their division and disjunction. Of the (mostly analytical) judgments, in which these categories of the relations between concepts are expressed, the judgments of subordination especially have acquired a rather dangerous significance from the fact that (as far back as Aristotle) Logic has given way to the temptation of regarding the subject thus conceived as falling within the sphere of the predicate as the type of all judgment, and subordination or subsumption as the prevailing meaning of the copula. This is an error in principle of the scholastic logic. "Gold is a metal" is indeed a real subordination; but "gold is yellow" never means in living thought that gold ought to be subsumed under yellow, which would be obviously nonsense—and certainly not always that gold is to be reckoned among yellow bodies, but rather that gold has the property of yellowness. Subsumption may be thought of as a side issue, but it is not the precise meaning of the judgment. But neither is it always predication, and even Aristotle's more careful way of proceeding from the content of the concept, making the predicate

a mark of the subject, is in many cases not correct. Such a proposition as: "Pure gold is sometimes found" falls under neither of the two schemata.

Our discussion of these relations was necessary to enable us to discover the principle which must govern our attitude to the *Syllogistic* which is founded upon them. From our point of view we may say that its task is to determine the forms under which *one concept holds good of others.* Even in determination, division and disjunction we could not dispense with the defining significance which the generic concept possesses in thought for all parts of its logical denotation (that is, for the generic and singular concepts which are subordinate to it); but this comes out still more generally in the fact that every determination which is valid of the generic concept, as such, is also valid for every part of its empirical denotation. The concept "implicitly contains" all its exemplars, it "represents" them, and its entire denotation may be substituted for it. Thus the fundamental logical relation of the dependence of the particular on the universal is verbally expressed by saying that every conceptual judgment can assume the form of the universal or the apodeictic proposition: S is P,—all S's are P,—every S must be P; hence we may be excused from any closer discussion here of the difficulties which may arise for the *dictum de omni et nullo* out of the ambiguity of the negation of the universal judgment.

It is only by reference to these linguistic matters that we can understand the theory of inference as it has been current since the time of Aristotle. It confines itself to the relations of sameness and difference which obtain between the *connotation* and the *denotation* of concepts; that is to say, it only takes into consideration one kind of reflective relations and passes by the constitutive categories with complete unconcern. The perfection to which Aristotle was able to develop the syllogistic system, into the details of which we need not enter, was due to this limitation. It is easy to show that the "false subtlety of the four syllogistic figures" and all their modes can be traced back to the influence of language. That, indeed, is abundantly evident in the reducibility of the remaining figures to the first, *i.e.* to the subaltern conclusion, which is precisely the one that expresses most clearly the dependence of the particular on the universal. Under it belongs also the so-called inference through opposition, as soon as the negative relation in the conclusion is rightly

conjoined with the principle of consequence; but this requires closer discussion, for which we have here no space.

The last and in a certain sense justifiable effect of this limitation of the theory of inference to the relation of equality between the connotation and denotation of the concept is to be found in the attempts, renewed in modern times, at a logical calculus. In conformity with the schematization of inferences by circles or angles—apparently favoured even in antiquity—and following the erroneous opinion that the copula signifies in some way an equality of subject and predicate, thinkers have always returned to the idea of writing judgments as equations, and finally of treating them as mathematical equations. The one feasible way of doing this, since in the Quantity of the judgment the extent of the subject was already determined, was that of *quantifying the predicate*: obviously it was then possible to compare the denotations on both sides and to reckon with them. Still it is clear from the above how narrow is the circle of relations to which the inference is thus limited: of the real inferences of living thought this theory of inference only applies without over-refinement to mathematical ones. Here once more we arrive at the result: there are logical principles of Mathematics but no mathematical principles of Logic.

The sphere of the *constitutive categories* coincides with that of the forms of thought which Kant claimed for his *Transcendental* as against Formal Logic, namely objective relations; and it was his great creative genius which taught us that this objectivity exhibits itself for our actual perceptual life in relation to *experience*, and is restricted to experience. Kant, it is true, was also of the opinion that his (objective) categories in themselves are as valid for all thinking whatsoever as the "analytical" forms, which come under the principle of contradiction. Hence the transcendental significance of these categories, should, in the last instance, be independent of the conditions of any and therefore also of our perception. But these categories owe that objectivity, with which we are here mainly concerned, precisely to their schematization in temporal and even partly in spatial forms. Herein lies the paramount significance of *schematism* in the Transcendental Logic. For what Kant represents as immersion in the sensuous schemata of space and time is in fact so essential for the objective relations that it may be regarded precisely as the quality which makes a distinct

species of this series of categories. If we exclude this mark we shall have only a formal logical relation, a reflective category: in place of the category of being (existence) we should once more have that of validity.[1] This relation is most illuminative and most characteristic in causality: if we strip it of its temporal character we are left with nothing but the general form of dependence or determination, the fundamental principle of which is the dependence of the particular on the general:[2] this is Spinoza's timeless or mathematical causality.

Accordingly I regard the principle according to which the series of categories are related to be as follows: since the reflective relations (sameness and difference) are thought as existing in the objects, they are coloured by determinations from the temporal and, to a certain degree also, from the spatial order. Time and space, then, have this place in Logic, that they turn reflective into constitutive categories. We may mention here, in answer to the many ill-founded views which have sprung up, out of psychological prejudices, around the critical separation of the "intuitive" from the "logical" forms, that the constitutive category is in itself a single and inseparable unity in which the "intuitive" and the "logical" stand for the two sides, which can be separated from one another only by abstraction. Every one of the perceptions which make up our experience contains a manifold of sensuous qualities ordered into a unity; but this order is never merely of a time-and-space character, but is always at the same time categorical: and these two orders are not even so bound up with one another that each one might exist for itself. They form an

[1] We must not, therefore, reproach Kant because he sought in the relation of the categories to the kinds of judgment a ground of common principles for the two parts of his Logic—the formal and the transcendental: we must hold firmly to the inner connexion between the two. The defect of the "transcendental analytic" is only that the "table of judgments" is "raked together" entirely historically. For the division is neither derived nor derivable from the essence of the judgment, but was taken over empirically from the scholastic Logic and trimmed up into a symmetrical trichotomy. Kant himself indicated the right relation when he again and again distinguished the categories, as in themselves empty thought-forms, from their objective application tc time-and-space perception; cf. for example, *Kritik der reinen Vernunft*, *Abschn.*: *uber Phaenomena und Noumena*, 1 Ed. p. 241, *Ab. Ausg.* iv. 158ff.

[2] Cf. *ibid.* p. 243 (iv. 159. 24), 2 Ed. p. 301 (iii. 206. 10). Generally this state of things is brought out especially clearly by Kant in the relation of the categories to principles: hence the difference in the formula employed in the two editions of the *Kritik der reinen Vernunft* is most significant and instructive.

inseparable unity which is intuitive-categorical, and which is on that account and only on that account the objective formulation of the manifold content. In the empirical movement of ideas it may happen on occasion that either the one or the other order occurs as fundamental and as giving the rule for apperception: but for objective knowledge they stand and fall together. That temporal successions form the occasion and handle for our insight into causal connexions is a purely methodological, not a logical relation. It is from this point that we get the simplest and clearest view of the Hume-Kantian problem.

By means of these relations the constitutive categories group themselves into the two series which may be called, after their chief representatives, those of *Substance* and *Causality*; because these, with Kant too, in the "Categories of Relation" and in the "Analogies of Experience" are the two fundamental moments. But let us call the existential aspect which is present in the constitutive categories, sometimes Reality (*res*) sometimes Actuality (*Wirken*). The mediation by the reflective categories is, under these presuppositions, comparatively easy to grasp. Existing sameness is called *identity* in so far as a plurality of more or less similar ideas are objectively related to that which is numerically one: and difference is called *change* in so far as a plurality of more or less different ideas are related in a similar way to a numerically single object. In the first case the persistence of the same, in the second the change in time of the different is accentuated. It thus becomes clear why identity can only be attributed to that which is distinct and change to that which is identical. The persisting identity is the *thing*, the relation of which to its different and changing properties constitutes the category of *inherence*.[1] And when the change consists in a happening, which may be analysed into a becoming and a vanishing, it becomes clear why it only takes place *between the states of a thing*, and signifies either an immanent happening in one and the same thing or a transeunt happening between the states of different things. The categorical unity in happening is *activity*, which implies the *necessity of temporal succession*: it is either causal, when the antecedent determines the consequent so as actually to exist in time, or it is teleological, when the result is regarded as that which determines its own conditions. (And here we may remark that end and

[1] We are here not considering the question whether the co-existence of the manifold, which goes with inherence, requires the multi-dimensional intuition of *space*.

motive belong to the causal and not to the true teleological forms of happening.) Necessity, then, is either a sequence or a demand. The same distinction also applies to the complex things that arise in the course of events, and which themselves in turn form things of a higher order: here either the whole is determined through its parts as mechanical product or, conversely, the part through the whole as organism or, as Driesch has called it, individual. We must not spin out these relations any further here: we must only emphasize the fact that the question of the application of these different kinds of constitutive categories falls within the sphere of Methodology.

In conclusion, however, we must call attention to the point that all possibilities of change must somehow reside and be grounded in the persistent nature of the thing. This is usually expressed as the relation of the attribute to its modes and as that of force or faculty to activities and states. But if we look more closely at this relation we shall find that it is that of the universal to the particular, *i.e.* that that reflective relation of determination has here become constitutive. The universal is one of the moments which bring about the particular. In addition to this, if the *dictum de omni et nullo* only seems, from the strictly logical point of view, to signify that the class notion is valid for its entire logical denotation, we find living thought quite accustomed to regard it as the measure of all its empirical content and determining all its actual instances. And here, we may remark in passing, the sense of the so-called general negative judgment passes over from conceptual impossibility to real exclusion. Finally, different events occur together in all circumstances (even though the occurrence be unique in character and is not repeated); their concurrence is regarded as necessary only because the temporal relation is determined by a *universal rule.* We call this universal rule a (causal or teleological) *law* and in this highest and concluding category we conceive a universal, which *is valid* for the particulars subsumed under it not only reflectively but also *constitutively*, although in reality we cannot have the faintest conception of the reality of such "generic concepts of changes" and of their real relation to that which takes place as conditioned by them. Neither Nominalism nor Realism applies here; but even pure Logic leads us, through this structure of the system of categories, to the view that validity and being, however carefully they must be separated from one

another, yet in the last instance cannot be entirely exclusive moments.

III. METHODOLOGY.

Strictly speaking, Methodology has no principles of its own. Its principles are to be found in pure Logic, and Methodology has only to deal with their application to the different aims of the special sciences. So far, it is a *technical discipline*, and might also be called the *organon* of the sciences or the doctrine of the *systematic* forms of thought. We must note one matter of principle, namely that every method must be defined only by reference to the particular logical and actual character of the subject-matter to which it is to be applied; so that as the sciences progress in actual insight they have to complete, improve, refine and extend their methods. This development must, of course, be left to the particular discipline itself. Logic has neither the right, the duty nor the power to excogitate fertile methods; and Methodology has only to see that, while taking the most concrete possible survey of the work of the special sciences, their problems and methods of procedure, it makes clear the different applications which are thus made of the logical forms and norms as they are brought together to serve some particular end. Hence Methodology must always remain the most incomplete part of Logic. In the development of Logic up to the present it has been the almost universal practice, in the tasks just described, to make use of the determinations of Formal and Analytic, to the exclusion of those of Transcendental, Logic.

This seems fully justified so far as concerns the universal aspect of Methodology, which deals with the *methods of proof and of refutation* which are equally valid for all the sciences and also for extra-scientific thought. For all these are only more or less complicated ways of inference and therefore have their principles in syllogistic. We are, however, obliged to pass beyond this formal schematization as soon as we reflect on the character of the *major premisses* which, themselves not demonstrable, must as immediate certainties form the starting point of all proof. Here, from the formal standpoint, the *quantity* of the judgment will be decisive methodologically; for the major premisses are either *axioms*, *i.e.* general presuppositions, which cannot be grounded in experience or *facts* which are given in perception. It is on this (Aristotelian) ground that the distinction between

rational and empirical sciences is based. But *Mathematics* alone of all the sciences can attain the ideal of grounding its proofs on axioms: it is the sole purely rational discipline. And if we call the remaining special sciences *empirical* we must not imply that they are based exclusively upon facts, but merely that we can work up these facts so as to be suitable for the purposes or knowledge only with the help of axiomatic presuppositions. As a rule the less these disciplines in their living work of knowing bring expressly into consciousness the axiomatic structure of their proofs which they take as a matter of course, the more necessary is it for Logic to work it out as systematically as possible. For, as I have shown in the *Präludien* (II. 108), it is the special task of Philosophy to establish, by empirical reflexion on the functions of universally valid evaluations, the norms whose actual axiomatic validity can indeed be illustrated by facts and brought to consciousness in them, but can never be proved by them. The method of Philosophy, then, is neither rational nor empirical, but *critical*; and Formal Logic must above all confirm this by its example (cf. above, p. 26 f.).

The two kinds of proof, of which one proceeds from the universal to the particular, and the other from the particular to the universal are known under the names of apodeictic and epagogic, or *deductive* and *inductive* methods. But the deductive method of proof plays a part not only in rational, but also in empirical science. For the syllogistic advance is possible when the general premisses differ very much in value. They need not be axioms in the strictest sense: they may also consist of defining determinations or of hypothetical concept- and judgment-constructions; or finally, they may be more or less certain results of inductive thought. And since the truth of the conclusion is conditioned by, that is, is dependent upon the truth of the premisses, the results of the deductive proof are only in the first case apodeictic; in the second they are, on the contrary, problematic, and in the third only probable. In all cases, however, this way of arriving at a particular from a universal proposition of some kind of real or assumed validity, is called *a priori deduction*; and Logic ought not to allow any other meaning of this much abused term to exist or to come into being. It is, however, plain from the above that in the empirical sciences also not only special demonstrations but often quite mportant facts are and must be entirely deductive or *a priori*

in character. As examples of the first case we may quote those discussions in which, from physical or chemical hypotheses, the consequences by which they can be verified are deduced, and for the second case large portions of systematic Jurisprudence.

The formal conditions of *Inductive proof* are also to be found in *syllogistic* reasoning. In this respect it amounts to the logical process of *reduction*: a universal proposition is to be based on singular or particular judgments, which syllogistically would follow from it by subalternation. This inference, which is in itself formally illicit, must in each instance be justified by a logical relation between the judgments already established and the one which is to be established. This holds good even of so-called *perfect induction*, which is by no means a mere shorthand abridgement, but establishes a universal (conceptually universal) judgment through the corresponding *general* judgment, which is empirical and comes by enumeration. There is no way of justifying this transition in principle except as in the case of *imperfect induction*, where we have to show that the ordering of individuals or species under the generic concept which is assumed in the premisses can be proved in no other way. The exclusion of collateral causes and the *probability* that the different cases have nothing in common except the generic character and the predicate corresponding to it in the concluding proposition, may be confirmed in very different ways—*inter alia*, by a very large number of premisses. But a large number of instances is never in itself a proof: under certain circumstances (*e.g.* in experiment) a single case may suffice for the induction, as long as it satisfies (as a so-called "pure" case) the logical demands of inductive proof. The theory of the latter must not therefore be confused with the theory of *probabilities*, which means something quite different; for it rests upon and also results in numerically determinable disjunctions. The "probability" of the inductive proof, on the other hand, means holding a thing true on insufficient grounds (cf. above, p. 34 f.), and the kind and degree of proof which is contained therein have to be determined.

The ultimate presupposition of induction is always the postulate *of the uniformity of natural law*, and this in the sense not only that the same causes produce the same effects but also that the same effects have the same causes. But such a simple reciprocity, which is at once causal and teleological (cf. above,

p. 43 f.) can only be assumed subject to considerable limitations; it is open to objections on many grounds. For this reason induction with all its expedients, such as analogy, etc., is in the last instance little more than a method of inquiry, the results of which only attain to full certainty when they coincide with that of a deductive proof from valid premisses. But the important point brought out by this analysis of inductive inference is that the logical significance of the conclusion turns out to lie not only in the formal moment which belongs to syllogism but in a *constitutive category, i.e.* causality. Out of this grows the task of providing a *methodological theory of inference* to complete the (formal and reflective) doctrine of syllogism—just as Kant intended his Transcendental Logic to supplement the Formal Logic. Contributions towards this end are to be found in Hegel's "Subjective Logic" and also in J. Stuart Mill's and Lotze's doctrine of causal inferences. But this work has never yet been taken in hand systematically and from all points of view.

We are led still deeper into the particularity of the objects of perception by the logical analysis of the *methods of investigation*: and if the first and universal part of Methodology refers us to Formal Logic the second part looks at once to Epistemology, since by the light it throws upon scientific investigation as a whole it corrects the *naïve* idea of the relation of knowledge to its "*object*." For the deciding principle, which dominates everything here, consists in this, that the objects of knowledge are never immediately given as such but have to be produced by every science for itself by means of a synthetic conceptual construction. It is comparatively easy to see this, and it is currently accepted in the case of the one purely rational science, Mathematics. That it produces its magnitudes by a synthetic construction of concepts and not by any sort of copying from experience may be regarded, since Kant's time, as one of the most secure and self-evident of doctrines. Moreover, in Mathematics, the relation of knowledge to its own self-constructed objects is of a highly instructive transparency. For though it may be a matter of choice in any inquiry as to what particular numerical forms or spatial figures shall be taken as its objects, yet once the object is constructed, the thought cognizing it is entirely bound to it and to the object's immanent conformity to law. The formation and solution of problems can proceed from nothing other than the development of the relations of magni-

tudes created by the synthetic construction itself. Mathematical thought, however free it is in the production of its objects, experiences just that *coercion of objectivity* which is essentially contained in the forms it has constructed and which victoriously opposes its sovereign power against every caprice of assertion. We will call this relation, which from the standpoint of subjective Psychology must appear as the wonder of wonders, the *Logic of the object.*

The same is true of all the *empirical sciences*, but it is concealed by the pre-scientific ways of thought which are characteristic of *naïve* Realism. Hence logical theory must insist all the more strongly that perception can relate itself to no other objects than those it has itself conceptually determined. It is an illusion to believe that there can ever be any experience whatsoever in which reality as such is taken up or reproduced. The truth is rather that, just as in non-purposive perception, only a very limited part of what is experienced comes into consciousness, so even the first logical work of turning a perception into a concept implies a selection and reconstruction of perceived moments (cf. above, p. 37); and this process is continued in all the farther constructions of conceptual thinking. In the discursive working out of generic concepts scholastic Logic calls this the inverse relation of the growth of the connotation and denotation of concepts. And similarly, in all other cases, the combining functions of scientific inquiry depend on free selection within the given material, and on a creative synthesis in the new disposition of its moments. But here, too, the "Logic of the object" reigns supreme; for though the direction followed in selection and synthesis be determined by the goal towards which the inquiry is consciously marching, yet the results of this new creation are equally determined by the essential necessity immanent in the objects so produced.

Methodology, which is indeed a kind of comparative morphology of science, has accordingly to inquire what are the principles in the different disciplines, by which the selection and synthesis in the production of objects is effectuated. If, in our search after these guiding threads, we proceed first of all according to the formal marks, we shall be met again by the quantitative opposition between the universal and the singular. In this connexion we have to distinguish between those sciences which are governed by laws and those which deal with events,

between nomothetic and ideographic inquiry. It is this which really makes the difference in intellectual interest between Natural Science and the Humanities. But we cannot repeat too often that we are here only speaking of ultimate aims and hence of those sciences which appear as polar opposites, between which the real work of Science moves in manifold gradations, so that in any particular case we can only speak of a preponderance of one or the other moment—as Rickert, in his penetrating analysis of this relation, has pointed out. The ultimate goal of all investigation of Nature is to attain timeless generic concepts of being and happening, but that does not exclude the fact that the way thereto leads over stages of simpler inter-connexions in which it rests and provisionally halts. For it is precisely in the real that the nomothetic rationalization of Reality must find its limits. On the other hand, the specific object of all historical inquiry is a construction which is significant chiefly because it can never recur, and which has to be lifted out of its entanglement in the non-significant elements lying all around it. To understand such a construction, however, History requires general concepts and axioms, which she is certainly more likely to borrow successfully from general experience than from the natural sciences (to which, from this point of view, Psychology also belongs); and it creates for itself the possibility of characterizing this unique object by a peculiar kind of generic concept and by a comparative study of the conformity of events to law. Thus generalizing and individualizing thought inter-penetrate: the one needs the other, and the methodological character of the particular science will decide which of the two shall serve it as end and which as means.

But the same fundamental distinction develops itself also at other actual moments. First of all we must remember that Methodology treats of *human* sciences, and that, therefore, in it normative Logic must be expressly drawn from anthropological data. Hence it now appears that the investigation of Nature (since, in dealing with perceptions, it is essentially concerned with the formation of generic concepts and the discovery of laws), possesses a *purely theoretic* and *trans-anthropological principle* of selection and synthesis. That the application of this principle in the empirical pursuit of Natural Science is determined partly by human needs and interests concerns only

the direction and sphere of the inquiry, and not its scientific procedure. As to its method, Natural Science is disinterested; hence the temptation, to which it has gladly succumbed, of considering and calling itself science κατ' ἐξοχήν. History, on the contrary, far from taking any event of whatever kind as its object, is much more inclined to place man as a being conscious of value as the central point of its selection and synthesis. The science of humanity deals with that in himself and his environment to which man has given value. How far trans-anthropological values have broken through and encroached does not concern History as empirical science; that is a matter for Ethics as a philosophical discipline. Methodology, grasping the special characteristic of historical science, finds that the pre-scientific beginnings of "narrative" depend on many empirical and all too human valuations, which creep into historical science itself. Methodology must show that the final ground for the general validity of the sciences of humanity lies in the universally recognized values which Ethics extracts from the movement of human life in History. But, in any case, historical investigation is *science related to values.* This means, first and foremost, that for it consciousness of value is the objectifying principle of selection and synthesis. But we must not, on that account, think that its opinions must consist of judgments of value, or that this methodological view has anything whatever in common with the ethics of desire and volition. This has been so clearly set forth and so expressly emphasized by Rickert, who was the first to recognize and elaborate this moment to a systematic perfection, that it ought not to be necessary to repeat it here.

We shall best provide against such misunderstandings if we consider a third moment which is contained in the difference between natural and historical investigation: it is concerned with the two different types of happening and of causality of which we spoke in the doctrine of categories (cf. above, p. 41). Natural Science analyzes perceptual constructions into their elements, and isolates these by real or ideal division, by experiment or analysis, in order to study the behaviour of the particular in its obedience to law. Physics, Chemistry and Psychology do this, each of them in its own way. On this account, however, the reconstruction of experience in these disciplines takes place according to the principle of *mechanical causality.*

This means that the complex physical and psychical facts are so conceived that the whole is regarded as the result of its parts and as entirely determined by them. But this way of constituting the object is inadequate within the sphere of historical knowledge. For with whatever it may concern itself—whether with persons and their aims, and the actions resulting therefrom; or with nations with their languages and political states, their customs and laws, their societies and religions, their arts and sciences—its objects are always those personal or supra-personal unities which have the structure that we call organic and in which the whole determines the parts just as much as the parts the whole. These are methodological distinctions of far-reaching significance, and are grounded in actual differences in the things known. When, therefore, the outlines of the latter become blurred by subtle transitions, the scientific methods of treatment will exhibit similar characteristics. This is actually the case in the biological sciences. Here the *décomposition des phénomènes* only suffices for the descriptive disciplines; and it is only the historical moment, the history of evolution, which promises to shed light on the facts of morphological co-existence. On the other hand, the theory of evolution can, in the strictest sense, attain to historical significance only if it introduces into the gradated scale of living beings, which appear for it as "higher" and "lower," relations of value of many kinds. In another direction there grows out of these relations the problem, which we can here only touch upon, because it, as it were, unites all the difficulties of Methodology *in nuce*—how far can psychical life be grasped by Psychology which, with its mechanical *causality of association*, follows the methods of the natural sciences?

Speaking generally, we may say that the fundamental thought-relation between the universal and the particular in the three fundamental forms of the special sciences may be characterized as follows: For mathematical intuition it is a relation, independent of all questions of origin, between the parts and the whole, and here it is entirely a case of relations of magnitude. For natural science the universal is the *abstract* concept or the law by which the particular is determined in its being and becoming: here the particular is explained when it is recognized as a special case of the universal. The sciences of humanity are everywhere concerned with the category emphasized by Hegel of

the *concrete universal*, which goes out from itself as a living unity and instances itself in the particular: here the individual is understood when it is recognized as a necessary constituent of a significant whole.

We can now, without detriment to the above-mentioned distinction of principle between the empirical sciences, establish a number of methodological forms of investigation which may be said to be common to both kinds of sciences, in the sense that they present analogous kinds of thought-movement. Here again we must remark that, so far, Logic has developed the theory of these on the side of the natural sciences rather than on the historical side. In marking out its sphere of investigation and in its first orientation therein every science, which is not merely a special branch of an already existing science, must start from pre-scientific ideas and the opinions or knowledge already contained therein. For this purpose the provisional conceptual determinations, which are what we ought to mean by the term *nominal definitions*, and the provisional *divisions* for which all kinds of schematic disjunctions are generally at hand, may be used. The customary formal demands on both of these need not here be set forth in detail: they may be satisfied without any lasting guarantee of scientific serviceableness. It is of much more importance in this respect that investigation corrects, justifies, limits, extends, adds and changes in many ways; and only as the result of the whole investigation do we get *real definitions* and *classifications*, the systematic significance of which is certainly not the same for all the different disciplines.

This whole progress, however, from the provisional to the definitive, is carried on by means of the collecting, ordering and categorical manipulation of *facts*—all processes which exhibit a purposive selection and re-combination of experiences. It is in this way that *naïve* perception becomes methodically transformed into *scientific experience*. The methods applicable depend, of course, upon the specific nature of the objects under investigation, and their development upon the degree of knowledge of these objects already attained. The more thoroughly knowledge has already penetrated into the essence of its objects, the more delicate and secure are the methods it can devise for its own perfectioning. Hence science grows quantitatively and qualitatively in geometrical progression, true to the fundamental

principle of all spiritual development that "to him that hath shall be given." The methods of *observation* and *experiment*, which the natural science of the external senses has in this way elaborated, have for their aim partly to extend or subtilize human sense-organs in their functioning, partly to isolate the object, and to subject it to a quantitative determination of measure. It is true that it is not possible either to determine objects numerically or to contrive the observation of them (*e.g.* by experiment) to the same degree in the different branches of natural investigation. On the contrary, *inner perception* which forms the foundation of Psychology can make hardly any use of these advantages. This science must, therefore, by the thoroughness of its conceptual analysis, make the more use of those which it derives from the constant and general accessibility of its fundamental facts. From physiological or so-called psycho-physical investigations it must expect but little help, and that only in its most elementary investigations. The human sciences, too, although certainly much later than the natural sciences, have created for the unambiguous and universally valid determination of their facts a widely ramified technique of critical procedure which makes it possible for many investigators to work together on a given plan. The very fine and thoughtfully developed rules and instruments which, in some cases, stand in rich variety at the disposal of the *criticism* and *interpretation* of tradition, have up to the present not been sufficiently elaborated on the side of their logical structure. I am inclined to find the reason for this, apart from the general direction which logical interests formerly followed, in the great intrinsic difficulty of the subject. For the significant and rational inter-connexion of facts, which always in such cases forms the ultimate pre-supposition, is perhaps much less exactly determinable logically than even the universal conformity of Nature to law which forms the major premiss of all scientific empiricism. Hence there must always remain for historical investigation a last moment which it is never possible to formulate methodologically, and which lies in the *intuitive* apprehension of those utterly different personal or supra-personal living wholes of which we have already spoken.

We can now understand that the establishment of facts in science presupposes not only the whole apparatus of its methodological instruments but also at every stage of inquiry

the results of knowledge already acquired. Hence, in the strict sense of the word, there are no purely *descriptive sciences*; these are at most the preliminary stages of *theoretical* disciplines. Even the pre-scientific "description" in which a perceived object was to be expressed, had to use words whose general meaning was familiar. These words correspond most nearly to the undefined general ideas of the pre-scientific consciousness: and when these, by means of logical thinking, are transformed into concepts (cf. above, p. 37) in order that they may acquire the clearness and definiteness requisite for scientific usage, the comparison of many perceptions is necessary. The particular in itself alone has never scientific validity. Moreover, the critical establishment, whether in natural science or in historical inquiry, of everything that cannot be immediately experienced, demands a knowledge of the *genetic interconnexions* which exist between the facts, the knowledge of which constitutes the essence of *explanatory theory*. From all these reasons we see not only that the description and confirmation of facts are the basis of explanation, but are only themselves possible to any degree of perfection by means of the latter.

Every inquiry, then, inevitably makes assumptions; and these assumptions, though they have co-operated in the establishment of the facts, are yet, in the last instance, to be tested as to their correctness by means of them. That which in demonstration would be forbidden as a "vicious circle" is in inquiry a sanctioned and fruitful auxiliary. Just as analytic geometry assumes that a problem is solved, in order to be able, by construction, to derive the conditions of the solution, so too empirical science works with presuppositions which she can only prove by the consequences to be afterwards developed therefrom. Thus, *e.g.*, inductive inquiry may lay down problematically a general axiom in order to deduce from it by subaltern inference analogous cases, from the actual existence or non-existence of which it can either be proved or refuted. Thus, too, philologico-historical hermeneutics starts from the construction of a significant whole in order to fill out the lacunae or to correct corruptions in what has come down to us. Hence the *logic of hypothesis* is the most important part of the Methodology of inquiry; and here again the logical structure has, up to the present, been worked out much more clearly for Natural Science

than for History. In both spheres, however, a distinction has to be made between *particular* and *universal* hypotheses. The former involve the assumption of particular, not directly perceived events; the second, general conceptual determinations as to the essence of things and their ways of acting. The proof of the hypothesis may in the first case, under certain circumstances, follow from supplementary perceptions (observation or experiment), and is then called *verification*. In all other cases it exhibits once again a *process of reduction* in which it is shown that the observed consequences can be entirely deduced from the assumed grounds and cannot be deduced from any others.

The *knowledge of reality* of the empirical sciences, then, consists in this: out of the endless mass of perceptions which are never entirely unifiable in the human consciousness it builds up, by means of carefully planned selection and synthetic combination, more or less comprehensive conceptual interconnexions, which are causal or teleological in structure. In this sense it possesses *immanent truth* in *the agreement of the theory with the facts.*

IV. THEORY OF KNOWLEDGE.

What is taught by the sciences has, as opposed to the opinions and convictions of individuals or of single groups of men, *objective universal* validity; and this universal acceptance logical theory must not disturb, but must rather acknowledge unconditionally. The only question which remains to be considered in view of these results is concerned with the presuppositions of the pre-scientific consciousness: how is this generally valid knowledge related to the reality to which it is referred as its object? The answer involves a *revision of the naïve identification of object and reality*, or of the relation of objective thought to reality or, in the last instance, of the relation between *consciousness and being*. For those, then, who would regard "Logic" as nothing more than the art of right thinking this question and the whole investigation directed towards its solution is indeed *metalogical*: and, since it is impossible to speak of the relation of consciousness to being and of thought to reality without speaking of being and of reality themselves, the Epistemological problem and inquiry are also *ontological* or *metaphysical*. Indeed we must say that in Critical Philosophy the Theory

of Knowledge has, with regard to the content of the problem, entirely superseded the old Ontology and Metaphysic. All the more clearly, however, must the methodological distinction be emphasized. The Theory of Knowledge, according to the Critical Philosophy, does not profess to possess or assert any knowledge of its own of absolute reality: it borrows its arguments in support of its attitude to these problems entirely from the sciences themselves, whose existence it unconditionally recognizes. On the other hand this must not be taken as a defence of feeble attempts to put together a "Metaphysic" out of the so-called general results of the sciences. The critical method of the Theory of Knowledge is concerned rather with the general question: What do the sciences themselves teach us by their activities and their theories as *to the relation of knowing to reality*?

If we are thus referring the ultimate and most difficult problems, in fact the final tasks of Theoretical Philosophy in general, to Epistemology, we can in this sketch only attempt to indicate the most important points of view from which pure Logic and Methodology have enabled us to treat such problems: the actual elaboration of the Theory of Knowledge or Critical Metaphysics can thus only be indicated in its most general lines.

First of all, that fundamental relation of which the theory of knowledge has to treat demands a more exact specification of the two terms which are to be related to one another. By "consciousness" we must not understand psychical state or psychical activity, still less a psychical being, a "subject," as bearer of psychical states or activities: for all these are themselves something real and belong to that to which being or existence is attributed, *i.e.* to the existent. What is meant, when such a question is asked, is rather the *content of consciousness*, that which is ideated or thought; and this too must be further specified, for we are here only concerned with the relation of objective, that is, universally valid thought to being. So too with the other members of the relation under discussion, namely, being; what is meant here is not that categorical relation which constitutes the fundamental significance of all constitutive categories (cf. above, p. 41), but something to which being is attributed.

If, then, the establishment *of the relation between that which*

is objectively thought and that which exists is the general formula for the problem of Epistemology, we must pass in review the different possibilities of its solution offered by the *doctrine of categories*: for there must always be some category or other by means of which such a relation is expressed and asserted in a judgment. But here again, as in Methodology, we must rid ourselves of the preconceived idea that this category must be the same for all kinds of knowing, that is to say, for all sciences. Such a preconceived idea is just as injurious and productive of error in the Theory of Knowledge as in Methodology, which has often suffered from its attempt to discover a universal method for all sciences, and with this object in view to force upon the other disciplines some one method which has been found valid in its own particular sphere. The *autonomy of the special sciences*, which rests on the difference of their objects, must make itself good by means of the Logic of the Object, not only in the uniqueness of its procedure but also in the specific colouring of the sense it attributes to the truth which it claims for its results.

The very slightest comparison of Mathematics with the other sciences warns us to take this precaution. For, first of all, our formula of the relation between what is thought and what exists seems absolutely inapplicable to *Pure Mathematics*; and hence Kant (for example), in his *Critique of Pure Reason*, and in the *Prolegomena*, only treated of Mathematics as applied to objects of experience, *i.e.* as an integral part of the theory of Natural Science. And, on the other hand, it must not be forgotten that the truth of the propositions which pure Mathematics lays down, either in the Theory of Numbers or in Synthetic Geometry, is entirely independent of any reference to a "reality" in the sense of the empirical sciences. Nevertheless, as already stated (p. 46), the Logic of the Object is to be found even in Pure Mathematics. Kant, in a most instructive passage in his *Prolegomena* (§ 38), has pointed out that we cannot help "attributing a nature" to a geometrical thing, such as a circle; and he then goes on to show, from the point of view of Transcendental Idealism (again, it is true, dilating on Mathematics as applied to Natural Science), that the obedience to law in which this "nature" of geometrical constructions consists can only spring from the "understanding," which determines space (for the behoof of the

particular concept) by means of its construction and according to the conditions of synthetic unity. There is no doubt that this is true of Pure Mathematics: here the coercion of objectivity in the process of knowing seems, as a matter of fact, to be given only in the nature of the intellect, in virtue of which it is obliged to draw all the conclusions which actually follow from the constructions it has itself made. But this actual, immanent Necessity of the object is something which exists in itself and which determines the activity of the psychical process over against which it stands, and it is only this immanent necessity which makes it possible to distinguish in mathematical thinking between truth and error. Here again that which is thought (*das Gedachte*), if it claim acceptance as generally valid and true, relates itself to an independent something, existing for itself, even though this cannot be regarded as "real" or "actual," in the usual sense of the word.

And here we have hit upon the central difficulty of the whole theory of knowledge. For those habits of thought which have been perpetuated in speech tend to represent that existent-in-itself which is the indispensable presupposition even for the kind of knowing we get in Pure Mathematics, as in some way or another a Real and to treat it as such. And manifestly the "nature" attributed to the "circle" (not to travel beyond Kant's) example is not nothing. It is a *something* and, moreover, a something whose existence and essence is quite independent of whether or not a knowing consciousness takes it for its object. It furnishes the rule for the correctness or incorrectness of the empirical consciousness which comes to a knowledge of itself in dealing with it. Such a something exists, then, and determines knowledge, and yet it has neither substantial nor functional reality or actuality. There only remains for us, therefore, either to ascribe to it a "being" peculiar to itself, which cannot be thought by means of any of the above-mentioned categories, or else to recognize and to find a name for its existence and its normative significance for knowledge as something peculiar, essentially different from all "being." Related if not altogether coincident lines of thought may be traced in antiquity: on the one hand, Plato ascribed to ideas a "real being," an ὄντως ὄν; on the other, the Stoics put side by side λεκτόν and ὄν as the two highest kinds of τί, their most general category. But this

is no mere question of terminological distinctions. For the first of these two paths necessarily leads to the metaphysical theory of two worlds, to the assumption of a higher existence (*Sein*) different from the empirical. Such a metaphysical "being," or "over-being," however, can have, logically, no other content than the "archetypes" of the (empirical) existent; and the kind of existence which it has, if it is not (as thing-in-itself) to be entirely undetermined, must be that of consciousness (*Bewustsein*), that is, it must be psychical. This has been the historical result of the ambiguity of the ἀσώματον in Platonism, and of the "super-sensuous" in Kant. Indeed, apart from this transmutation into spiritual reality, that "super-existent" or "over-being" (*Übersein*) threatens to become what cannot in any wise be presented in thought, and has therefore been more and more eliminated from the modern Theory of Knowledge and Metaphysic. If we follow this line of thought, however, all being acquires the character of the "sensuous" and empirical, since the "sensuous," according to the Kantian terminology, includes consciousness as psychical, as the determination of the inner sense. The shades of meaning which the constitutive categories, or the forms and the relations of being acquire through the intuitive moments of time and space, are in complete accord with this view.

All the more necessary, then, is the second way out, which Lotze followed when he coined the term "validity." But the convenience of the expression, which this happily chosen word supplies, does not absolve us from the task of fixing exactly its significance in logical use. In the first place we must exclude the psychological sense of the word, which means more or less actual recognition, the fact that it is believed on the part of the empirical consciousness. But we must also exclude the normative significance of that-which-ought-to-be-believed, in so far as this is supposed to include within itself in its entirety the postulate of general consent. Both of these only contain the secondary relation, whether normative or actual, between that which is valid and the knowing consciousness; and just for that reason they presuppose the valid-in-itself as an element independent of the movements of the empirical ideational process. But the nature of this constituent which can determine the judgment while, as the theory of categories shows, it also acquires a certain amount of constitutive significance (and this is also

true of the validity of mathematical relations for the theory of Nature and for Nature itself)—this kind of existent, which is yet not to involve any "being," forms the chief metalogical problem of the theory of knowledge. It is in the main the same difficulty which has existed from the beginning in the interpretation of Kant's consciousness-in-general, which also does not claim to be, and ought not to be, conceived either psychologically or metaphysically; although, as we may see from Kant himself, the difficulty of avoiding this double danger is extreme.

That which remains to us after the exclusion of all the metaphysical and psychological misinterpretations of both these terms, of "the valid" as well as of "consciousness-in-general," can, in my opinion, be nothing else than the sum-total of the *inter-connexions and relations* between existents. They are not themselves existents, either as things, as states, or as activities; they can only become "actual" as the content of the psychical functions of knowing. But in itself the *realm of the valid* is nothing else than *the form and order under which that which exists is determined.* These forms are valid in themselves, they are valid for that which exists and they are valid for the process of knowing; but their validity for being and for knowing is only grounded in the purely essential validity which belongs to them in themselves. The existent does not bring about this form and order, nor does knowing produce them; but as there is no being which does not exist in this form so there is no process of knowing which does not make use of it. The relation between the valid and the existent holds good for all Mathematics, geometrical or arithmetical, and it also holds good of all the purely logical categories, whether reflective or constitutive. For even the laws of thought of the reflective series of categories have been found to be grounded in postulates which signify that the existent is subject to form and order (cf. above, p. 29 ff.).

This relation between validity and existence, which is no other than that of *form* and *content*, is the ultimate and irreducible point, beyond which the analysis of knowledge cannot pass. That order of the existent which validity means is no stranger to being itself, although it appertains neither to that which is contained in it nor to that which is derived from it, but to something in it which is movable and which is akin to

it. Hence an explanation of this relation would refer us to a still higher point of union, above validity and being, and is therefore altogether impossible. But we here catch sight of the reason why *metalogical speculation* can take no other path than that of a *spiritualistic Metaphysic.* If we want to make the impossible possible, to conceive the forms of validity which equally determine being and knowing, as themselves in their turn something existent and active, there remains nothing left to us, since they are given to us as objects of our knowing, that is to say, in psychical actuality, than to think the formal structure of the valid as a "spiritual order" and to connect it with a spiritual first principle. The "central Monad" of Leibniz, Berkeley's "God," Kant's *Intellectus archetypus,* Fichte's *Ego,* and Hegel's "Idea," are all attempts, assertorical or problematical, to satisfy this want. But we must at least remember, if we try to conceive validity as a kind of psychical being or over-being, that between such a world-ordering spirituality and our human spirit there is about as much similarity as obtains, to speak Spinozistically, between *Canis signum coeleste* and *canis animal latrans.*

This comes out clearly in the fact that, for human knowing, there is yet another relation between validity and being: *i.e. the discrepancy between form and content.* All the ultimate problems of the old Ontology and Metaphysics came to this, that the presuppositions which are contained in the demonstrative and determining forms of consciousness are never completely satisfied in the data of experience. The conceptual reconstruction which, *e.g.*, B. Herbart regarded as the essence of philosophy, has precisely as its task to transform the *naïve* but inadequate connexion between form and content till these render each other complete mutual satisfaction. But this *antinomianism* can never be eradicated from the nature of human thought, and it seems only to indicate that the forms grounded in the general and all-pervading elements of reason can never come to their own either perfectly or purely in the fragmentary and superficial content of human experience. Hence the *construction of objects* in human experience and science can never be more than *provisional,* and the attempt to think pure form objectively must necessarily lead to its complete emptying of all content and hence to its uselessness for the intellectual mastering of experience. Thus, within the category of inherence lies the presup-

position of persisting identity: but none of the "things" of perception do this full justice. Hence science supersedes them by the concept of substances as true "real" things; and since these concepts prove to be dependent on the constructional needs of the various disciplines, *e.g.* Physics, Chemistry, Organology, Psychology, this line of thought ends in the concept of *thing-in-itself.* But as the idea of "thing-in-itself" has no content whatsoever, the synthesis of the manifold which it signifies becomes dissolved and, in the strictest sense, objectless. A similar tragedy in the progress of knowing may be seen when we ask what is meant by *happening.* The further we pass beyond experience and isolate the elements of the causal connexion, the more the living activity is lost, in which, after all, the synthetic interconnexion between antecedent and consequent consists. Finally, in this antinomian insufficiency of existence as opposed to validity we may remind ourselves of the fact which was significant even for Plato, namely, that mathematical relations are never perfectly realized in that which is. We need only point here to the analogous and more palpable antinomies of the ethical and aesthetic consciousness. Everywhere in the human life of reason that which is super-human and universally valid appears as embedded in a being and happening which indeed accommodates itself to it but can never completely assimilate it. *Προθυμεῖται μὲν πάντα τοιαῦτα εἶναι οἷον ἐκεῖνο, ἔστι δὲ αὐτοῦ φαυλότερα* (Plato, *Phäd.* 75 b).

But let us recall the fact that the different ways of solving the problem of Epistemology depends upon the doctrine of categories; for the relation between objective thought and that which exists is always determined by some category or other. We shall find that all the way from *naïve* to advanced scientific thought it is the category of Likeness (*Gleichheit*), a category of reflexion, which rules. It is taken as the fundamental condition of our apprehension of *transcendent truth.* This is the standpoint of *naïve Realism*; the world is as we perceive it.

But I need not here show in detail how this way of looking at things, so far as it relates itself to our knowledge of external nature, has been driven out step by step by the empirical sciences. If my view is right there remains only inner perception in which likeness, as of a copy or image, is

accepted as the criterion of truth for memory reproducing the actual experience. Here alone, where the actual fact is itself consciousness, is the knowledge of the fact either just the fact itself or an immediate repetition and image of it; that is to say, an image which no intervening medium has distorted or obscured. It is only Psychology which can comply with the demands of this *naïve* conception of Truth; and even it can do so only under strict limitations, methodically deduced from the nature of memory; for memory underlies even the laws of the ordering, selecting, completing and transforming apperception. But as to our perceptual knowledge of the things of the external world, reflective thought has long been accustomed to regard it from the standpoint of the ancient Symbolism. Its elements are univocal signs which, while they operate upon the perceiving consciousness, are not therefore to be regarded as copies of the things themselves. This view applies the category of causality to the relation between knowing and being, and we can understand how it was that its final result was to put the serviceability of signs and of combinations of signs in the place of their truth. The simple semeiology of earlier times took the theoretical significance of this view to be that the data of sensuous perception do not exist as such really, but are only presentations or, as they were then called, ideas. This is the historical meaning of the term *Idealism*, which ought never to have been confused with other significations. It holds, not that existence must be denied to sense-data, but that psychical existence must be attributed to them; and this —in spite of common opinion—derogates so little from their value that a Lotze can say that the blossoming in consciousness of these effects of things is unspeakably more valuable than anything which may take place between things.

But the likeness of thought and thing can after all be only partially, never totally, annulled: for that which is thought as the empirical content of consciousness itself always remains (as is historically witnessed by the transformation of all Idealism into Spiritualism) an existent. Hence, in this direction, lie a multitude of epistemological possibilities which are distinguished from one another by the fact that they bring different strata of ideas under the symbolic standpoint. While elementary sensualistic Realism accepts the data of sensuous perception as

real, and, in accordance with the Nominalistic prescription, regards all concepts and logical relations as merely the effects of these upon consciousness, Rationalistic Realism conversely combines the doctrine of the subjectivity of the sense-qualities with the view and the purpose of converting real things into similar concepts. It conceives them either according to the *Mathematical Realism* of scientific theory under the purely quantitative determinations of space, time and motion; or with the *Ontological Realism* of the old metaphysic under purely categorical relations—but in both cases alike it proceeds in accordance with the "dogmatic postulate" that the world is as we are obliged to think it.

While I leave it to the reader to draw out the schema of the categorical possibilities of the Theory of Knowledge, finding a place among them for such historical curiosities as Positivism, Solipsism, etc., I shall bring forward one more version of the relation between thought and existence. The symbolic and idealistic theories have been prone to substitute for the relation of cause and effect, in which they had their historical beginnings, that of *Reality* and *Appearance* (*Wesen und Erscheinung*), which had another meaning, within the series of the categories of inherence. "Phenomenon" (appearance) means, on the one side, the way in which one thing is presented to another, but, on the other, the way in which existence exhibits itself in its states and conditions. There would therefore be no objection to Phenomenalism if, combining the two meanings of the word, it represented the content of consciousness in human knowledge as an appearance in which the nature of being after a certain fashion exhibits itself. But modern Philosophy for the most part does not draw this conclusion; on the contrary, it accentuates, one-sidedly and deliberately, that reality and appearance are not the same. It seems to be accepted almost as an axiom that the real must be *different* from its appearance and that therefore the qualitative determination of the latter must not be carried over to the former. The Idealism of Natural Science, which attributes none of the qualitative contents of perception to "real things," but determines the latter entirely quantitatively, has adopted this difference between reality and appearance as a principle; and when, later, the spatial and temporal moments, and even the psychical states, were included in appearance, there was absolutely nothing left for the thing-in-itself (which had to

be something quite different from the appearance) of the content experienced. Thus Phenomenalism became Agnosticism. But the difference between the thing-in-itself and the phenomenon can never be established: it ought, therefore, never to be posited assertorically but only problematically. Nor does it follow from the causal relation: for this relation holds good whether cause and effect are like or unlike.

The critical revision of this fundamental problem of the Theory of Knowledge brings us therefore to the question whether, among the different points of view of the natural sciences, there are any arguments which compel us to regard the existence which can be experienced and scientifically dealt with as the "appearance" of a higher existence, of an over-being which is a thing-in-itself and therefore unknowable. So far as I can see the affirmative answer to this question is not supported by *theoretical* arguments: the limitations of human knowledge are rather to be found in another direction. All scientific knowledge, as Methodology shows, presents a *piece cut out of Reality*, which, as synthetically complete, never really exists. We must, in this sense, attribute not existence but value to the object: still it contains pure moments of being and it includes them in an interconnexion which is valid for that object. No description of facts can ever completely comprise or imitate the reality with which it is concerned: but it can weave the selected constituents into a form which agrees with their real interconnexion. This is true, however, in the highest degree, of theories. The generic concepts and the laws of Natural Science are of course abstractions which, as such, cut off from all particulars, do not "exist": but they comprise all these particulars, they hold good for them, they are the order or system in which the actual nature of the things stands. Finally, the human sciences abstract from the endless mass of events interconnexions which, as they present them to us—aloof from and unaffected by all the other things round about them—have never come to pass. And yet these interconnexions, which represent the "object" of historical investigation, bring out the significance and value which lay within that which actually happened.

Thus, out of the universe, out of the physical and historical cosmos, each of the sciences forms in its "object" a little world which is only a piece, but is still a piece, of that

great world. We have no reason to believe that "behind" these particular worlds of knowledge a still higher one is concealed, from which they, as its appearances, differ entirely in content. But between the fragments which we thus elaborate into knowledge there must be other fragments, other "worlds" which have so far remained closed to our outer and inner perception, and perhaps to our selective synthesis. We are sure of this, because the particular worlds which are accessible to the special sciences, one and all at their boundaries point beyond themselves.

For such a *selective* theory of knowledge the "unconditioned," in the original sense of Kant's transcendental dialectic, is nothing more than the "totality of the conditioned." It does not therefore require the μετάβασις εἰς ἄλλο γένος, by means of which the thing-in-itself was transformed to a qualitative Other. But this totality still retains enough unknowability to exclude all Metaphysic in the old dogmatic sense of the word.

We men must content ourselves with this work of knowledge, fragmentary as it is. For it is our own original work—a new creation of the spirit as much as are Art and the moral life. It is true that the interconnexions whose validity, we are certain, reaches beyond our human nature, imply the postulate of a last unifying interconnexion of all reality. But this whole is closed to our knowledge: we shall never know more than a few fragments of it, and there is no prospect of our ever being able to patch it together out of the scraps that we can gather. The categorical relation of knowledge to being, of that which can be investigated to that which is beyond investigation, is not that of appearance to thing-in-itself, but of the *parts to the whole.* Absolute reality is not something qualitatively other than the being we know, but the one living whole of which we can only hew out pieces to make into our world of knowledge. But this whole, if we may think it under a category, is a self-articulated organism, which cannot be spelt out from its parts (cf. above, p. 41 f.). Hence modern Metaphysic, with its attempts to piece itself together out of borrowings from the sciences, is far more contemptible than the old Ontology which, starting from the realm of validity, had at any rate the courage to attempt the deduction of the interconnexion of the universe as an articulated whole. We have

seen that this is denied to man. All that we can do is to go on building up, by steady work, the particular worlds of knowledge over the construction of which we are masters, in the hope expressed in Goethe's words: *Nun, man kommt wohl eine Strecke!*

THE PRINCIPLES OF LOGIC

BY

JOSIAH ROYCE.

SECTION I.

THE RELATION OF LOGIC AS METHODOLOGY TO LOGIC AS THE SCIENCE OF ORDER.

§ 1. A VERY frequent account of the office of Logic runs substantially as follows: "Logic is a Normative Science. It deals, namely, with the Norms whereby sound or correct thinking is distinguished from incorrect thinking. It consists of two parts,—a general part, called Formal Logic, which defines the universal or formal normative principles to which all correct thinking must conform, and a special and very extended part called Applied Logic, or Methodology, which deals with the norms of thought in their application to the methods used in various special sciences."

From this conventional account the present sketch will deliberately depart. A discussion of some of the more important problems of Methodology will be comprised in our first section. The remaining paragraphs of this paper will be devoted to indicating, very summarily, the nature of a doctrine of which the traditional General or Formal Logic is but a part, and, in fact, a very subordinate part. To this doctrine the name "The Science of Order" may be given. It is a science which is indeed incidentally concerned with the norms of the thinking process. But its character as a normative doctrine is wholly subordinate to other features which make it of the most fundamental importance for philosophy. It is to-day in a very progressive condition. It is in some notable respects new. It offers inexhaustible opportunities for future progress.

§ 2. Everyone will agree that throughout its history Logic has been concerned with the conduct and with the results of the thinking process. Now the thinking process is indeed, from its very nature, *methodical*. In every human science, and in every human art that is teachable at all, the thinking process appears either as the creator and the guide, or else as the formulator and the analyzer, of the methods which characterize this science or art. If an art grows up instinctively, as the product of social need and of individual talent, the efforts to teach this art, so that it may pass from master to apprentice, lead sooner or later to an analysis and thoughtful formulation of the methods employed by the skilful workman. And when an art or a science is deliberately invented or advanced by the conscious skill of the individual inquirer or discoverer, the procedure used either includes a purposeful application of already known methods to new undertakings, or else involves an effort to create new methods. Everywhere, then, the consciousness of method grows in proportion as thought comes to play a successful part in the organization of human life.

Since, however, the methods used vary with the different arts and sciences, and yet have certain important features that are common to all or to many of the undertakings of these arts and sciences, it is natural that a comparative study of methods should form the topic of a more or less independent body of doctrine. And, as a fact, such a Methodology, such a "Normative doctrine," such an effort to survey and to systematize the methods used by all, or by one or another great body of thoughtful workers, has repeatedly constituted the principal task assigned to Logic, whether the distinction between General or Formal Logic and Applied Logic has been emphasized or not. Logic as a branch of philosophy began, as is well known, when the differences of opinion amongst the various philosophers, when the dialectical problems brought to notice by the Eleatic school, and when the more or less practical inquiries of the Sophists into the arts of disputation and of persuasion, had led to a conscious need for a general study of the methods of right thinking. In Aristotle's case the task of surveying, and in part of creating, a systematic body of sciences constituted an additional ground for undertaking a general methodology of the thinking process. And ever since Aristotle the view that one main purpose of Logic is to expound the "Art of Thinking," or the definition of Logic

in some other more or less exclusively methodological fashion, has played a large part in the history of our science. And this is why the definition of Logic as a Normative Science is still so common, and in its place useful.

As a fact, however, Methodology, taken in its usual sense as a study of the norms and methods of thought used in the various arts and sciences, is the mother of Logic taken in the other sense hereafter to be expounded. For the undertakings of Methodology lead to certain special problems, such as Plato and Aristotle already began to study, and such as recent inquiry makes more and more manifold and important. These problems, when considered for their own sake, assume an aspect that pretty sharply differentiates them from the problems of Methodology proper. They are problems regarding, *not* the methods by which the thinker succeeds, nor yet the norms of correct thinking viewed as norms, but rather the *Forms*, the *Categories*, the *Types of Order*, which characterize any realm of objects which a thinker has actually succeeded in mastering, or can possibly succeed in mastering, by his methods. Taken in this sense, *Logic is the General Science of Order*, the *Theory of the Forms of any Orderly Realm of Objects*, real or ideal.

Just because Logic, viewed as such a doctrine, has resulted from the efforts to formulate the norms and methods of thinking, the question how Logic as Methodology differs from and yet gives birth to Logic conceived as the Science of Order, must be summarily indicated in the rest of our opening section. To this end, we must consider some of the principal problems of Methodology.

§ 3. Let us then first return to a brief mention of some of the problems of method which characterized the well known early stages of logical inquiry, as they are represented, for instance, in remarks that frequently recur in the Platonic dialogues.

The "plastic youth" of the Platonic dialogues, is to be instructed by Socrates in the right method of thinking, and is to be warned against the false arts of the Sophists. The instruction that he most frequently receives relates: (1) To the proper method of definition; (2) To the task of systematic classification, with the prevailing use of dichotomy for the sake of dividing a wider class into its constituent species; (3) To a careful study of the evidence which attaches to certain notable propositions; (4) To a watchful examination of modes of

inference. The special considerations which are so frequently repeated in the Platonic dialogues in regard to each of these matters, do not here, in any detail, concern us. It is enough to recall a few facts only. Definition, for instance, according to the Socratic and Platonic methodology, depends indeed upon a collection of special instances of the concept that is to be defined. But, as Socrates often points out, instances, taken merely as such, constitute no definition. For we do not learn what clay is merely by remembering or by naming several different sorts of clay. One must conceive, in universal terms, what is common to these sorts of clay. And so too it is if we want to define justice, or virtue, or knowledge. Definition gets at the essence, at the "Idea," at the type, which special instances exemplify, and depends upon taking the universal as such, and upon bringing it to our knowledge with clearness. But a definition, once thus formulated upon the basis of the instances first chosen, needs to be further tested. One tests it, according to this methodological doctrine, by applying it to new instances, and by a deliberate search for possible inconsistencies. For a truly universal account of a concept must provide for all the cases that rightfully fall under the concept which is to be defined, and must exclude all instances which do not belong to the type in question. In case inconsistencies are discovered, by finding that the definition includes too much or too little, the definition first attempted must be amended. But in such consideration of right definitions, one is greatly aided by remembering that no universal types exist in isolation. And here a very important feature of Plato's methodology appears. *The universals, the "Ideas," form a system.* There are the more and the less inclusive universals. Instances, or classes of instances, which appear to possess mutually inconsistent characters, may still be conceived as members of the same larger class, and in so far as illustrating the same universal, if only they can be shown to be determined to be thus distinct through a process of classification, whereby the essence of the more inclusive universal is in fact more clearly portrayed than it could be through a merely abstract definition. One knows number, in its universal essence, all the better, when one learns to classify the numbers as even and odd, as perfect squares or as not perfect squares, and so on. Such classifications are very commonly best made in the form of dichotomies.

The class A may be divided into the A that is *b*, and the A that is not *b*. Arrays of classes and sub-classes may be arranged by repeating such a process. And then a sub-class whose traits are very highly specific, may be defined in universal terms by considering, first A (some "highest genus," as, in terms of the later logic, we may already name it); then B, which comprises whatever A possesses the character *b*; then C, which comprises whatever B possesses the differential mark *c*, and so on. Thus definitions may be rendered both consistent and systematic, and the system or true Order of the universals may be at least approached, if not fully grasped.

As for the evidence which attaches to single propositions, that also must be considered in the light of special test-cases, must be subjected to the criterion of consistency, and must be made familiar by repeated examination. In the course of such examination and re-examination of the convictions which most interest the philosopher, the importance of a clear consciousness regarding the nature of correct inference often comes to light. One is clear that one infers rightly, not when one is carried away by the Sophist's torrent of persuasive oratory, but when one observes the necessity of each individual transition from thought to thought. If one believes that "All A is B," a closer examination readily shows the general truth that one may not thence infer that "All B is A." Yet in hasty discourse, or under the influence of a Sophist's oratory, one might let such a false inference pass unheeded.

§ 4. So much may here suffice as a mere hint and reminder of thoughts which now seem methodological commonplaces, but which, at that early stage of the history of Logic, were momentous for the whole future of the subject. The elementary text-books still repeat the substance of these observations, even if their context is no longer that which appears in Plato's dialogues.

It will be noted at once that such a methodology naturally leads to a view of the nature and constitution of the world of truth, whose significance, at least as Plato conceived it, goes far beyond the value of these precepts as guides for the learner of the art of thinking. If, namely, these things are so, then, in Plato's opinion: (1) *The realm of the Universals or "Ideas" is essentially a System*, whose unity and order are of the first importance for the philosopher; (2) *Inference is possible because truths have momentous objective Relations*, definable precisely in

so far as the process of inference is definable; (3) *The "Order and Connection" of our rational processes,* when we follow right methods, *is a sort of copy of an order and connection which the individual thinker finds, but does not make.* One thus sets out to formulate the right method. One discovers, through this very effort, a new realm—*a realm of types, of forms, of relations.* All these appear to be at least as real as the facts of the physical world. And in Plato's individual opinion they are far more real than the latter. Thus Methodology leads Plato to a new Ontology. The world of the Forms becomes the world of the Platonic Ideas; and Dialectic, with its methods, becomes for Plato the gateway of Metaphysics. Here he finds the key to unlock the mystery of Being.

We are not in the least concerned to estimate in this discussion the correctness or even the historical significance of the Platonic Metaphysic,—a doctrine thus merely suggested. It is enough to note, however, that even if one sets aside as false or as irrelevant all the principal metaphysical conclusions of Plato, one sees that in any case the Methodology of the logician, even in this early stage of the doctrine, inevitably gives rise to the problem as to the relatively objective order and system of those objects of thought to which the methodologist appeals when he formulates his procedure. The Platonic theory of Ideas, Aristotle's later theory of Forms, the innumerable variations of the Platonic tradition which the subsequent history of thought contains,—all these may or may not be of use in formulating a sound metaphysic. But in any case this comes to light: If a logician can indeed formulate any sound method at all, in any generally valid way, he can do so only because certain objects which he considers when he thinks,—be these objects definitions, classes, types, relations, propositions, inferences, numbers, or other "principles,"—form a more or less orderly system, or group of systems, whose constitution predetermines the methods that he must use when he thinks. This system, or these systems, and their constitution, are in some sense more or less objective. That is: What constitutes order, and what makes orderly method possible, is not the product of the thinker's personal and private caprice. Nor can he "by taking thought" wilfully alter the most essential facts and relations upon which his methods depend. If an orderly classification of a general class of objects is possible, then, however

subjective the choice of one's principles of classification may be, there is *something* about the general nature of any such order and system of genera and of species,—something which is the same for all thinkers, and which outlasts private caprices and changing selections of objects and of modes of classification.

Meanwhile (as we may here add by way of general comment), orderliness and system are much the same in their most general characters, whether they appear in a Platonic dialogue, or in a modern text-book of botany, or in the commercial conduct of a business firm, or in the arrangement and discipline of an army, or in a legal code, or in a work of art, or even in a dance or in the planning of a dinner. Order is order. System is system. Amidst all the variations of systems and of orders, certain general types and characteristic relations can be traced. If then the methodologist attempts to conduct thinking processes in any orderly way, he inevitably depends upon finding in the objects about which he thinks those features, relations, orderly characters, upon which the very possibility of definite methods depends. Whatever one's metaphysic may be, one must therefore recognize that there is something objective about the Order both of our thoughts, and of the things concerning which we think; and one must admit that every successful Methodology depends upon grasping and following some of the traits of this orderly constitution of a realm that is certainly a realm of facts.

§ 5. This brief reference to the consequences to which the Socratic and Platonic Methodology so early led, may suffice to suggest a deep connection between Methodology proper, and what we have called the Science of Order. This connection becomes only the more impressive if we pass from those elementary and now commonplace considerations which play their part in the methodological passages of the Platonic dialogues, to a few observations that a brief review of contemporary scientific thinking will readily bring to the mind of any fairly well informed student.

Let us then at once turn from the earliest stages of Logic to its latest phases. Let us here omit any attempt to expound the Aristotelian Logic, or to estimate its methodological value, or to tell its later history. Let us pass over the often repeated story of the Baconian reform of scientific methods and of the vastly more important consequences of the experimental methods which Galileo and his contemporaries introduced into modern

science. Let us come directly to the present day; let us remind ourselves of some of the most familiar of the doctrines of modern scientific Methodology; and then let us see how these doctrines also lead us to problems which demand their own special treatment, and which again force us to define a Science of Order,—a science distinct from Methodology proper, but necessary to a true understanding of the latter.

It is a commonplace of modern Methodology that our knowledge of nature is gained through induction, and upon the basis of experience. It is equally a commonplace that scientific induction does not consist merely of the heaping up of the records of the facts of crude experience. Science is never merely knowledge; it is orderly knowledge. It aims at controlling systems of facts. Amongst the vastly numerous methods which various sciences employ in our day, there are some which stand out as especially universal and characteristic means of accomplishing the aim just emphasized. Let us mention the most prominent of these methods. Such mention will at once bring us again into contact with the fundamental problems whose nature we are here attempting to illustrate.

And so, first, every science, in dealing with the facts of experience, employs *Methods of Classification*, and is so far still making its own use of the lessons that Socrates taught. There is, in the development of every new science of nature, a stage in which, in the absence of more advanced insight into the laws to which the facts are subject, classification is the most prominent feature of the science. Botany and Zoology, in their earlier stages of growth, were, for a considerable time, sciences in which classification predominated. Anthropology, in its treatment of the problems presented by the racial distinctions of mankind, is still very largely in the stage of classification; while in other of its fields of work, as, for instance, in its comparative study of the forms and results of human culture, Anthropology now pursues methods which subordinate classification to the higher types of methodical procedure. Amongst the medical sciences, Psychiatry is just emerging from the stage where the classification of cases, of symptoms, and of disorders made up the bulk of the science; and has begun to live upon a higher plane of methods. In the Organic Sciences the stage of classification (as such instances remind us) very generally endures long, and is with difficulty transcended. And the more complex the facts to be

understood, the harder it is for any science, organic or inorganic, to get beyond this first stage. In the case of Chemistry we have a notable instance of a science where the complexity of the facts long forced the science to consist in large part of the enumeration and classification of elements, compounds, properties, and reactions, despite the fact that the experimental methods used were especially well adapted to lead to a knowledge of very general and exact laws. Recent Chemistry, however, has grown far beyond the stage of mere classification.

Where a science passes from this early stage to one of higher insight, *two* more or less sharply distinct types of methods, either separately or (as oftener happens) in combination, frequently play a large part in determining the transition. These are (1) The type of the methods that involve *comparing the corresponding stages* in the various *processes or products of natural Evolution* with which the science has to deal; and (2) The Statistical Method proper, that is the method *which uses exact enumerations as the bases of inductions.*

§ 6. In the wholly or partly organic sciences, the Comparative methods just mentioned play a very large part. How they lead, beyond the stage of classification, to higher sorts of knowledge, is well exemplified by the case of Geology. That science began with classifications of rocks and of formations. But almost from the outset of the science it became evident that these formations were not sudden creations, but had been the results of processes that had required long periods of time. The earlier efforts of "Vulcanists" and "Plutonists" to furnish adequate universal theories of these processes in more or less simple terms, showed that other methods must be used. The key to unlock *one* portion of the mysteries which the new science was to explore, was furnished by the comparative study of the geological formations found in various regions of the earth's crust. When this comparison showed, for instance, corresponding series of fossil-bearing strata, a new light was thrown upon the history of the earth. To be sure, such comparative study of geological series of formations and of fossils, constitutes but one portion of the resources of Geology. Other methods, and very different ones, play their part in Dynamical Geology. But the importance of the comparative study of corresponding geological formations for Historical Geology, serves as one example of what makes the comparative method, in its various

analogous forms, significant in great numbers of scientific investigations.

Suppose, namely, that what is to be studied consists of the stages or of the results of any evolutionary process whatever. Something has grown, or has resulted from the ageing or from the "weathering" of the crust of a planet, or from the slow accretion of the results of a civilization. Rock formations, or the anatomical constitution of various organisms, or social systems such as those of law, or such as customs, or folklore, or language, are to be understood. One begins with classification. But herewith science is only initiated, not matured. For it is the evolutionary process itself, or the system of such processes, which is to be comprehended. The comparative procedure it is which first *correlates the corresponding stages of many analogous or "homologous" evolutionary processes and products*, and thus enables us not merely to classify but to unify our facts, by seeing how the most various phenomena may turn out to be stages in the expression of some one great process.

§ 7. Side by side with the Comparative Methods stand the Statistical Methods. These two sorts of methods are, in fact, by no means always very sharply to be distinguished. There are various transitions from one to the other. Every comparison of numerous evolutionary processes, or of the results of such processes, involves of course some more or less exact enumeration of the cases compared.

But such enumeration may not be the main object of consideration. Very many statistical enumerations are guided by the definite purpose to carry out with precision the comparative methods just exemplified. But, as the well known applications of statistical methods to insurance, and to other highly practical undertakings show us, the most characteristic features of the statistical procedure are independent of any such interest as leads the geologist to his correlations of corresponding formations, or the comparative philologist to his analysis of corresponding grammatical forms in different related languages. The Statistical Methods are often used as a short road to a knowledge of uniformities of nature whose true basis and deeper laws escape our knowledge. Mortality tables are good guides to the insurance companies, even when medical knowledge of many of the causes of death remains in a very elementary stage. The statistics of marriage and divorce, of suicide and of crime, or of

commerce and of industry, furnish bases for sociological research, even when there is no present hope of reducing the science in question to any exact form.

But whatever their uses, the Statistical Methods involve us in certain problems which have to do with the *correlation of series of phenomena.* A glance at any considerable array of statistical results serves to show us how the mere heaping up of enumerations of classes of facts would be almost as useless as the mere collection of disordered facts without any enumeration. Statistical results, in fact, when they are properly treated, serve to describe for us the constitution of objects whose general type Fechner had in mind when he defined his *Collectivgegenstände.* Such a *Collectivgegenstand* is a conceptual object which results when we conceive a great number of individual facts of experience subjected to a process of thought whereof the following stages may here be mentioned:—

(*a*) These individual facts are classified with reference to certain of the features with respect to which they vary. Such features are exemplified by the varying sizes of organisms and of their organs, by the various numbers of members which different interesting parts of the individual objects in question possess, by the extent to which certain recorded observations of a physical quantity differ from another, and so on.

(*b*) This classification of the facts with reference to their variations having been in general accomplished, the Statistical Method enumerates the members of each of the classes, in so far as such enumeration is possible or useful.

(*c*) The various enumerations, once made, are arranged in orderly series, with reference to questions that are to be answered regarding the laws to which the variations in question are subject. Such series, in case they are sufficiently definite and precise in their character, tend to show us *how two or more aspects of the phenomena in question tend to vary together,*—as, for instance, how human mortality varies with age; how the mean temperature of a place on the earth's surface varies with its latitude or with the season of the year; how the size of an organ or an organism varies with conditions that are known to be determined by heredity or by environment; and so on.

(*d*) *Various series,* when once defined with reference to such features, *are correlated with one another,* by means which the

Methodology of the various Statistical Sciences has further to consider.

(*e*) And, as a result of such processes, the statistician comes to deal with "aggregates" or "blocks" of facts which, taken as *units*, so to speak, *of a higher order*, appear as possessing a structure in which laws of nature are exemplified and revealed. Such *ordered aggregates treated as units of a higher order are Collectivgegenstände.*

Now it is obvious that every step of such a methodical procedure presupposes and uses the concepts of *number*, of *series*, and of the *correlation of series*; and that the whole process, when successful, leads to the establishment of *an orderly array of objects of thought*, and to the revelation of the laws of nature through the establishment and the description of this order. *The Concept of Order is thus a fundamental one both for the Comparative and for the Statistical Methods.*

§ 8. Both the Comparative Methods and the Statistical Methods are used, in the more developed sciences that employ them, in as close a relation as possible to a method which, in the most highly developed regions of physical science, tends to supersede them altogether. This Method consists in *The Organized Combination of Theory and Experience.* This combination reaches its highest levels in the best known regions of physical science. Its various stages are familiar, at least in their most general features. But the methodological problems involved are of great complexity, and the effort to understand them leads with peculiar directness to the definition of the task of the general Science of Order. Let us briefly show how this is the case. In order to do so we must call attention to a familiar general problem of method which has so far been omitted from this sketch.

By the Statistical and by the Comparative methods, laws of nature can be discovered, not with any absolute certainty, but only with a certain degree of *probability*. The degree of probability in question depends (1) upon the number of instances that have been empirically observed in applying these methods, and that have been compared, or statistically arrayed, and (2) upon the fairness with which these facts have been chosen. Since every induction has as its basis a finite number of empirical data, and in general a number that is very small in comparison with the whole wealth of the natural facts that are

under investigation, any result of the comparative or of the statistical methods is subject to correction as human experience enlarges. A question that has always been prominent in the discussion of the general methodology of the empirical sciences, is the question as to our right *to generalize from a limited set of data*, so as to make assertions about a larger, or about an unlimited set of facts, in which our data are included. By the Comparative Method, one learns that such and such sets or series of facts are thus and thus correlated,—as for instance that the geological strata so far observed in a given region of the earth's surface show signs of having been laid down in a certain order, with these and these conformities and non-conformities, faultings, foldings, and so on. How far and in what sense has one a right, by what has been called "extrapolation," to extend the order-system thus defined to more or less nearly adjacent regions, and to hold that any still unobserved geological features of those other regions will be, in their character and order, of the type that one has already actually observed? Or again, by the Statistical Method, one learns that certain facts enable one to define a *Collectivgegenstand* of a certain type. How far can one rightly "extrapolate," and extend one's statistical curves or other statistical order-types, to regions of fact that have not yet been subject to enumeration? For instance, how far can one make use of mortality tables, framed upon the basis of previous records of death, for the purpose of insuring lives in a population which inevitably differs, in at least some respects, from the population that has already met with its fate, and that has had its deaths recorded in the mortality tables?

The general answer to this question has often been attempted by methodologists, and has usually taken the form of asserting that such "extrapolation" logically depends, either upon the principle, "*That nature is uniform*," or upon the still more general principle: "That *every event*" (or, as one sometimes asserts, "*every individual fact*") "*has its sufficient reason.*" It is commonly supposed, then, that the basis of our right to generalize from a limited set of data to a wider range of natural facts, some of which have not yet been observed, may be stated in either one of two ways:—(I.) "These and these facts have been observed to exemplify a certain order-system. But nature is uniform. That is, nature's various order-systems are all of them such as to exemplify either one invariant type, or a certain

number of definable and invariant types. Hence the type of the observed facts can be, with due generalization, extended to the unobserved facts." Or again, using the so-called "Principle of Sufficient Reason," one has often stated the warrant for extrapolation substantially thus: (II.) "The facts observed are such as they are, and conform to their own order-system, not by chance, but for some Sufficient Reason. But a sufficient reason is something that, from its nature, is general, and capable of being formulated as a law of nature. The facts still unobserved will therefore conform to this same order type (will exemplify this same law), *unless there is some sufficient reason why they should not conform to this type.* This reason, if it exists, can also be stated in general terms, as another law of nature, and must in any case be *consistent* with the reason and the law that the observed facts have exemplified. Since law thus universally reigns in the natural world, since all is necessary, and since the observed facts not merely are what they are, but, for sufficient reason, *must be* what they are, we ought to regard the laws in terms of which the observed facts have been formulated as applicable to unobserved facts, unless there is a known and probable reason why they should not so conform. To be sure, our conclusion in any one case of such extrapolation is only probable, because it must be admitted, as a possibility, that there may be a sufficient reason why at least some of the unobserved facts should conform to laws now unknown. But the presumption is in favour of extrapolations unless sufficient reason is known why they should not be attempted."

§ 9. Familiar as such modes of stating the warrant for generalizations and extrapolations are, it requires but little reflection to see that the formulations just stated *leave untouched the most important features of the very problem that they propose to solve.* Let us suppose that one who is, in regard to a given scientific field of investigation, a layman, hears the expert give an account of certain uniformities of the data that have been observed in the field in question. So far, of course, the layman is dependent upon the expert for the correctness of the report. If the question then arises, "What right is there to generalize from these observed uniformities, so as to apply them to unobserved facts that belong to this same general field?" is the layman now able to use a general principle "That Nature is uniform," to decide this matter? No! The layman, if properly

critical, usually knows that this latter question is quite as much one for the expert to decide, as it is the expert's business to observe or to estimate the uniformities that have already come under observation in his own realm. In the geological case, for instance, the question whether or no certain special features of formations that have already been explored are likely to be repeated in regions not yet subject to geological study, is itself a question for the geologist. It cannot be settled by any appeal to the supposed general principle of the "Uniformity of Nature." That principle, in its abstract formulation, fails to help us precisely when and where we most need help.

Nature, in fact, is indeed full of uniformities. But what these uniformities are is itself a matter for observation. And only the very sort of experience that assures us of certain observed uniformities, can be our guide whenever we attempt to generalize from the observed uniformities to the unobserved ones. Sometimes the fact that certain uniformities have been observed, gives us very good warrant for expecting them to be repeated, in definite ways, in other regions of experience. Sometimes this is not the case, beyond some very limited range. Thus, the fact that a given man has lived ninety years, gives no presumption, based upon the general "uniformity of nature," that he will continue to live long in future. On the contrary, we are accustomed to say that, just because of "the uniformity of nature," as we now know it, he is likely to die soon; because, at his age, whenever an exceptional man chances to reach it, the general death rate is presumably high in proportion to the number of men of ninety years of age.

It follows that, if one uses the principle of the "Uniformity of Nature" as the basis for his extrapolations and generalizations, he has at once to face the question: "What uniformities are of importance in the field in question?" And to this question the *general principle of uniformity gives no answer.* This answer can only come from an empirical study of the uniformities that each region of nature presents.

Equally useless, in aiding us with reference to any one decision regarding our right to generalization and extrapolation, is the direct application of the "Principle of Sufficient Reason." How can we judge, in advance of experience, whether or no there is a "sufficient reason" why the facts not yet observed in a given field should agree in their order-systems with the facts

that have already been observed? Surely, by itself, the abstract "Principle of Sufficient Reason," even if fully granted, only assures us that every fact, and so, of course, every order-system of facts, is what it is by virtue of *some* sufficient reason, which is of course stateable in general terms as some sort of a law. But, the very question at issue is whether the still unobserved facts of any given field of inquiry conform to the *same* laws, and so have the *same* "sufficient reasons," as the thus far observed data. This question can admittedly be answered with certainty only when the now unobserved facts have come to be observed. Till then all remains, at best, only "probable." Now the "Principle of Sufficient Reason" does not by itself state any reason why only a *few* laws, or a *few* sorts of sufficient reasons should with probability be viewed as governing nature. It does not, therefore, of itself establish *any* definable probability why there should not actually be a sufficient reason why the unobserved facts should conform to new laws.

Thus neither the abstract principle of the "Uniformity of Nature" nor the still more abstract principle of "Sufficient Reason," serves to assure us of any definite probability that observed uniformities warrant a given generalization or extrapolation into regions not as yet subjected to observation. The question "What observed uniformities are such as to warrant a probable generalization in a given field?" is a question whose answer depends not upon any general application of either of the foregoing principles. They could both hold true in a world whose facts were such as defied our efforts to find out *what* the uniform types in question were, and *what* sufficient reasons there were for any fact that took place.

§ 10. What consideration is it, then, which makes generalizations and extrapolations, upon the basis of already observed uniformities, probable? To this question the American logician, Mr. Charles S. Peirce, has given the answer that is here to be summarized.[1]

This answer will especially aid us in understanding why the methods of comparison, and the statistical methods, inevitably lead, whenever they succeed, to a stage of science wherein the

[1] See Peirce's article on the Logic of Induction in the "Studies in Logic by Members of the Johns Hopkins University" (1883), and his article on "Uniformity," and several passages in his other contributions to Baldwin's *Dictionary of Psychology and Philosophy*.

method which organically unites Theory and Observation, becomes the paramount method. And hereby we shall also be helped to see why the types of Order whose methodical employment characterizes the highest stages of the natural sciences, are the proper topic of a special science that shall deal with their logical origin and with their forms.

Suppose that there exists any finite set of facts such as are *possible objects of human experience*, that is, suppose that there exists a finite set of facts belonging to what Kant calls the realm of *mogliche Erfahrung*. *One* presupposition regarding these facts we may here make, for the sake of argument, without at this point attempting to criticize that presupposition. It is the simple presupposition that these facts, and so the whole aggregate of them, whatever they are, have *some definite constitution*. That is, according to our presupposition, there are possible assertions to be made about these facts which are *either true or false* of each individual fact in the set in question. And, within some range of possible assertions which we here need not attempt further to define, it may be presupposed that: "Every such assertion, if made about any one of those individual facts, and if so defined as to have a precise meaning, either is true or is not true of that fact." Thus, if our realm of "objects of possible experience" is a realm wherein men may be conceived to be present, and if the term *man* has a precise meaning, then the assertion, made of any object *A* in that realm, "*A* is a man," either is true or is not true of *A*. And if our realm of objects is supposed to be one which consists of black and white balls deposited in an urn, the assertion, "*A* is a white ball," made about one of the balls in the urn, either is true or is false.

This presupposition of the *determinate constitution* of any set of facts such as are subject to inductive investigation, is by no means a simple, not even a "self-evident" presupposition. This, indeed, we shall later have occasion to see. But this presupposition, as Peirce has shown, is the one *natural and indispensable presupposition in all inductive inquiries*. And it is further Peirce's merit, as an inductive methodologist, to have made explicit a consideration which is implicitly employed by commonsense in the ordinary inductive reasonings used in the market place, or in any other region of our practical life. This consideration is that, *if we once grant the single principle of the determinate constitution of any finite set of facts of possible*

experience, we can draw probable conclusions regarding the constitution of such a set of facts, in case we choose "fair samples" of this collection, and observe their constitution, and then generalize with due precautions. And in order thus to generalize from the sample to the whole collection, *we do not need any presupposition that the collection of facts which we judge by the samples has a constitution determined by any further principle of "uniformity" than is at once involved in the assertion that the collection sampled has in the sense just illustrated, some determinate constitution.* In other words, given a finite collection of facts which has *any* determinate constitution whatever—be this constitution more or less "uniform," be the "sufficient reason" for this constitution some one law, or any possible aggregate of heterogeneous "reasons" whatever—it remains true that we can, *with probability*, although, of course, *only* with probability, judge the constitution of the whole collection by the constitution of parts which are "fair samples" of that whole, even when the collection is very large and the samples are comparatively small.

That we all of us make inductions, in our daily business, which employ the principle of "fair sampling," is easy to see. Peirce has emphasized the fact that the concept of the "fair sample" is not a concept which requires any special presupposition about the uniform constitution of the collection from which we take our samples. It is possible to judge by samples the probable constitution of otherwise unknown cargoes of wheat or of coal, the general characteristics of soils, of forests, of crowds of people, of ores, of rubbish heaps, of clusters of stars, or of collections of the most varied constitution. A mob or a rubbish heap can be judged by "samples" almost as successfully as an organized army or an orderly array of objects, if only we choose from the large collection that is to be sampled a sufficient number of representative instances. And the commercially useful samples employed when cargoes, or other large collections are to be judged, are frequently surprisingly small in proportion to the size of the whole collection that is to be judged by means of them.

§ 11. The reason why such a procedure gives good results can readily be illustrated. Let us take one of the simplest possible instances. Suppose that a certain collection consists of *four* objects, which we will designate by the letters *a*, *b*, *c*, *d*.

And to make our instance still more concrete, suppose that our collection consists in fact of four wooden blocks, which are marked, respectively, by the letters (*a*, *b*, *c*, *d*). Suppose that these blocks are precisely alike, except that they are painted either *red* or *white*. Let us hereupon suppose that somebody is required to judge how *all* the four blocks are colored, by drawing *two* of them at random from a bag in which they are concealed, and by then forming the hypothesis that, just as the colors *white* and *red* are present in the pair that he draws, precisely so these colors will be present and distributed in the whole set of four. In other words, if he draws two white blocks he shall be required to generalize and say: "All four of the blocks are white." If he draws one white and one red block, he shall be required to say: "Half of the blocks (that is, two of them) are red, and the others white."

Suppose next that, as a fact, the blocks *a* and *b* are red, while the blocks *c* and *d* are white. Let us consider what results of such a process of judging the four objects by a sample composed of two of them, are now, under the agreed conditions, *possible*. Of the four blocks (*a*, *b*, *c*, *d*), there are six pairs :—

(*a*, *b*) (*a*, *c*) (*a*, *d*) (*b*, *c*) (*b*, *d*) (*c*, *d*).

Six different samples, then, could be made from the collection of blocks under the supposed conditions. Of these six possible samples, One, namely, the sample (*a*, *b*) would consist, by hypothesis, of two red blocks. Whoever chanced to draw that sample, so that he was consequently required, by the agreement, to judge the whole set by that pair, would judge erroneously; for he would say: "All the four blocks are red." Whoever chanced to draw the pair (*c*, *d*), would have to say: "All the blocks are white." And he too would be wrong. But whoever drew any one of the *four* samples, (*a*, *c*) (*a*, *d*) (*b*, *c*) (*b*, *d*), would by agreement be obliged to say: "Two of the blocks are red and two are white," since he would be obliged, by the agreement, to judge that the whole collection of four showed the same distribution of white and red as was shown in the pair that he had drawn. Thus, if all the possible pairs were independently drawn by successive judges, each one drawing one of the possible pairs from the bag in which the four blocks were hidden then, under the supposed agreement, *two* of the judges would be wrong, and *four* of them right in their judgments.

This simple case illustrates the principle which Peirce uses in his theory of the inductive procedure. In general, if we choose partial collections from a larger collection, and judge the constitution of the whole collection from that of the parts chosen, fixing our attention upon definable characters present or absent, in the partial collections, we are aided towards probable inferences by the fact that there are *more* possible "samples," or partial collections, that at least approximately *agree* in their constitution with the constitution of the whole, than there are samples that widely disagree. Two of the possible samples in the foregoing simple case disagree, four agree, in the character in question, with the collection which is, by the supposed agreement, to be judged by the samples. That is, the possible ways of successful sampling are in this case twice as numerous as the possible unsuccessful ways.

What holds in this simple case holds in a vastly more impressive way when the collections sampled are large. Only then, to be sure, the probable inferences are, in general, only approximations. Suppose a large collection containing m objects. Suppose that a proportion r per cent. of these objects actually have some character q, while the rest lack this character. Suppose that the whole large collection of m objects is to be judged, with reference to the presence or absence of q, by some comparatively small sample containing n of these objects. The success of the judgment will depend upon how far the sample of n objects that happens to be chosen differs from or agrees with the whole collection, with reference to the proportion r' per cent. of the n objects which possess the character q. Of course it is possible that $r=r'$.

In case of large collections and fairly large samples, this will not often be exactly true. But if we consider *all possible selections* of n objects from the collection of m objects, even if n is a comparatively small number, while m is a very large number, a direct calculation will readily show that decidedly *more* of the *possible* sets of "samples" containing n objects will somewhat closely resemble in their constitution the whole collection in respect of the presence or absence of q than will very widely differ in their constitution from that collection. The matter will here in general be one of approximation, not of exact results. If, once more, r' per cent. represents the proportion of the members of a given sample of n objects that possess the

character q, while r per cent. is the proportion of the members of the whole collection that possess this same character q, it is possible to compute the number of *possible* samples consisting of n objects each, in which r' will differ from r by not less than or by not more than a determinate amount, x. The computation will show that, as this amount of difference increases, the number of possible samples in question will rapidly decrease.

In consequence, as Peirce points out, our inductive inferences can generally be stated thus, in so far as they involve the direct processes of sampling collections :—

"A proportion r' per cent. of the P's have the character q.
The P's are a 'fair sample' of the large collection M.
"Hence, *probably and approximately*, a proportion r' per cent. of the large collection M have the character q."

The ground for this probability thus rests, not upon the uniformity of the collection M, but upon the fact that more of the possible "fair samples" agree approximately with the whole than widely disagree therewith.

Now a "fair sample" of the large collection M is a sample concerning which we have *no reason to suppose that it has been chosen otherwise than "at random,"* or in a representative way, from among the objects of the large collection that we judge.

Thus the methodology of inductive generalization, so far as the statistical and the comparative methods are concerned, rests simply upon the principle that the facts which we study have a determinate constitution, to which we can approximate, with probability, by fairly sampling the whole through a selection of parts. From its very nature the procedure in question in all such cases is therefore essentially *tentative*, is subject to correction as comparison and statistical enumeration advance from earlier to later stages, and is productive of approximately accurate results, and, in general, of approximations only.

From this point of view we see why it is that experience may be said to teach an expert in a given field, not only what uniformities have been observed in that field, but what approximate and probable right one has to generalize from the observed to still unobserved uniformities in precisely that region of experience. For the process of sampling tends, in the long run, to correct and to improve itself, so as to show to the expert, although generally not to the layman, what ways of sampling are "fair" in their application to a given region of facts. For

the expert is one who has had experience of many samples of different *ways of sampling* in his own field.

§ 12. Herewith we are prepared to understand a step forward in methodical procedure which took place early in the history of physics, and which has since become possible in very various regions of science. It is obvious that such a step might be expected to consist in some improvement in the choice and in the definition of the regions within which the selection of "fair samples" should be made possible. As Peirce has pointed out, it is just such improvement that takes place when induction assumes the form of *sampling the possible consequences of given hypotheses* concerning the constitution or the laws of some realm of natural phenomena, *or of sampling facts viewed with reference to their relation to such hypotheses.*

The reasoning which is used when hypotheses are tested, is of a fairly well known type. The instance furnished by Newton's hypothesis that a falling body near the earth's surface and the moon in its orbit were alike subject to a force that followed the law of the "inverse squares," has been repeatedly used as an illustration in the text-books of the Logic of Induction. We need not here dwell upon the more familiar aspects of the method of the "working hypothesis" and of its successful verification, or of its correction in the light of observation. Our interest lies in the bearing of the whole matter upon the Theory of Order. This bearing is neither familiar to most minds, nor immediately obvious.

We must therefore sketch the general way in which the union of Theory and Observation is accomplished in the more exact natural sciences, and must then try to show that *what makes this union most effective, depends upon the possibility of defining hypotheses in terms of certain conceptual order-systems whose exactness of structure far transcends, in ideal, the grade of exactness that can ever be given to our physical observations themselves.*

In its simplest form, the method of induction here in question appears as a discovery of natural processes, structures, or laws, through an imaginative anticipation of what they *may be*, and through a testing of the anticipations by subsequent experience. The first and most directly obvious use of an *Hypothesis*, which thus anticipates an observable fact, lies of course in its *heuristic* value. It leads an observer to look

for what he otherwise might not have sought. It directs his attention.

But this, after all, is the least of the services which a good hypothesis renders to science. Its higher service is that, when it is indeed a good type of hypothesis for the field in which it is used, it may be made the starting point of a more or less extended *Deductive Theory*, which enables the investigator to discover indirect means of testing the hypothesis, in cases where direct means fail. One often meets with the remark that a scientific hypothesis must be such as to be more or less completely capable of verification or of refutation by experience. The remark is sound. But equally sound it is to say that a hypothesis which, just as it is made, is, without further deductive reasoning, capable of receiving direct refutation or verification, *is not nearly as valuable to any science as is a hypothesis whose verifications, so far as they occur at all, are only possible indirectly, and through the mediation of a considerable deductive theory*, whereby the consequences of the hypothesis are first worked out, and then submitted to test. If Thales successfully predicted an eclipse, he made and verified a hypothesis. But if this hypothesis was solely founded upon an empirical knowledge of the cycle of former eclipses, his astronomy had not yet passed beyond the statistical stage, and could not pass beyond that stage through even a large number of such verifications. But when a modern astronomer deals with lunar theory, and uses the comparison between theory and observation as, in this case, a very accurate means of testing the degree of accuracy of the Newtonian hypothesis of the law of inverse squares, as the law to which a field of gravitative force is subject, the value of the work done depends upon the vast range of deductive theory which here *separates* the original Newtonian statement from the observed facts. The recorded positions and movements of the moon, when supplemented by the records of the known eclipses that were recorded by the ancients, constitute a very vast "sample" of the physical facts about the moon's motion. The computations which lunar theory makes possible constitute a still vaster "sample" of special results of the Newtonian theory, as applied to the moon. Now if the Newtonian theory of gravitation had only a chance, or a temporary, or a superficial relation to the observable motions of the moon, the chances are extremely small that a very large sample of the

results of the theory should agree as nearly as they do with so large a sample of the results of observation. For in such a case *two* samples of facts, the one selected from a realm of *observed physical phenomena*, the other selected from the realm of *the ideal consequences of the Newtonian theory of gravitation*, are compared, not merely in general, but in detail; so that the correspondence of theory with observation is a correspondence of the two samples, so to speak, member by member, each element of each of the two samples approximately agreeing with some element of the other with which, in case Newton's original hypothesis is true, it ought to agree.

§ 13. What here takes place is, *mutatis mutandis*, identical with what constitutes the most important feature in any successful and highly organized combination of Hypothesis, Theory, and Observation. The stages of the process are these.

(1) A Hypothesis is suggested regarding the constitution or the laws of some region of physical fact.

(2) This hypothesis is *such as to permit an extensive and exact Deductive Theory* as to what ought to be present in the region in question, *in case* the hypothesis is true. *The more extensive, exact and systematic the theory thus made possible proves to be, the larger are the possible samples of the "consequences of the hypothesis" which are available*, whenever they are needed, for comparison with the physical facts.

(3) Samples of facts are chosen from a field open to observation and experiment, and are then compared with the results of theory. The more complete the theory, the larger the range of facts that can be called for to meet the need for comparison.

(4) This comparison no longer is confined (as is the case when the statistical and the comparative methods in their similar forms are used) to noting what proportion, r' per cent., of the members of a sample have a certain relatively simple character q. On the contrary, in case the deductive theory in question is highly developed and systematic, the sample of the results of theory which is accessible for comparison is not only complex, but *has a precise order-system of its own* (is, for instance, a system of ideally exact physical quantities) *which must be approximately verifiable in detail in case the original hypothesis is true.* The comparison of theory and fact is therefore here possible with a minuteness of individual detail which, in case of

successful verification, may make it very highly probable that if the system of real physical facts under investigation has any determinate constitution whatever, its constitution very closely agrees with that which the hypothesis under investigation requires.

It thus becomes obvious that the value of the method here in question very greatly *depends upon the exactness, the order, and the systematic character of the concepts in terms of which the hypotheses thus indirectly tested are defined.* If these concepts are thus exact and systematic, they may permit extended and precise deductions, and the result will be that large samples of the exact consequences of a hypothesis, will be such that they can be compared with correspondingly large samples of the facts of observation and experiment. The comparison of two such samples can then be made, not merely in general, but element by element, minutely, with reference to the Order presented and conceived, and in such wise as to make a chance agreement of theory and fact extremely improbable.

The result will be that the truth of the hypothesis that is tested will still be at best only *probable and approximate*, but the probability will tend to become as great as possible, while the approximation will grow closer and closer as the theory reaches more and more exactness and fulness of deductive development, and as it is confirmed by larger and larger ranges of observations.

An almost ideal union of deductive theory with a vast range of observations is found in the modern doctrine of Energy.

§ 14. In view of the foregoing considerations, we can now readily see that this, the most perfect of the scientific methods, namely *the organized union of Theory with Observation requires for its perfection concepts and systems of concepts which permit of precise and extended deductive reasonings*, such as the Newtonian theory of Gravitation and the modern theory of Energy exemplify. It is a commonplace of Methodology that hypotheses which are stated in *quantitatively precise terms*, especially meet, *at present*, this requirement, and lead to physical theories of the desired type. Our account, following Peirce's view of induction, shows *why*, in general, such theories are so important for the study of nature. The "samples of possible consequences" which they furnish are especially adapted to meet the requirements of a minute comparison, element by element, with the

samples of observed facts in terms of which the theories in question are to be tested.

Meanwhile our sketch of the general Theory of Order will hereafter show us that quantitative concepts get their importance for deductive theoretical purposes *simply from the fact that the Order-System of the quantities is so precise and controllable a system. Herein, to be sure, the quantities are not alone amongst conceptual objects*, and it will be part of the business of our later sketch to show that *the two concepts, Exact Deductive Theory and Quantitative Theory, are by no means coextensive.* The prominence of quantitative concepts in our present physical theories is nothing that we can regard as absolutely necessary. There may be, in future, physical sciences that will be highly theoretical, and that will not use quantitative concepts as their principal ones. Yet it is certain that they will use *some exact conceptual Order-System.*

But, however this may be, our result so far is the following one:—

A sketch of Methodology has shown, in the case of the Comparative, and the Statistical Methods, and of the Method which unites Observation and Theory, that all these methods use and depend upon the general concept of the *Orderly Array* of objects of thought, with its subordinate concepts of *Series*, of the *Correlation of Series*, and of special *Order-Systems* such as that of the *Quantities.* All these concepts are essential to the understanding of the methods that thought employs in dealing with its objects. And thus a general review of Methodology leads us to the problems of the Science of Order.

SECTION II.

GENERAL SURVEY OF THE TYPES OF ORDER.

§ 15. WHEN the methodical procedure of any more exact physical science has led to success, the result is one which the well known definition that Kirchhoff gave of the science of Mechanics exemplifies. The facts of such a science, namely, are "*described*" with a certain completeness, and in as "simple," that is, in as *orderly* a fashion as possible. The *types of order* used in such a description are at once "forms of thought," as we shall soon see when we enumerate them, and forms of the world of our physical experiences in so far, but *only* in so far as, "approximately" and "probably," our descriptions of the world of the facts of "possible physical experience" in these terms are accurate. The philosophical problem as to *how and why the given facts of physical experience conform as nearly as they do to the forms of our thought*, is a question that can be fairly considered only when the types of order themselves have been discussed precisely *as* forms of thought, that is as "constructions" or "inventions," or "creations," or otherwise stated, as "logical entities," which our processes of thinking can either be said to "construct" or else be said to "find" when we consider, not the physical, but the logical realm itself, studying the order-types without regard to the question whether or no the physical world exemplifies them.

That this mode of procedure, namely the study of the order-types apart from our physical experience, is important for our whole understanding of our logical situation (as beings whose scientific or thoughtful interpretation of nature is in question), is especially shown by the considerations with which our sketch of Methodology has just closed. For it is notable that *all highly developed scientific theories make use of concepts,—*

such for instance as the quantitative concepts,—*whose logical exactness is of a grade that simply defies absolutely precise verification in physical terms.* The Newtonian theory of gravitation, for instance, can never be precisely verified. For the conception of a force varying inversely with the square of the distance, with its use of the concept of a material particle, involves consequences whose precise computation (even if the theory itself did not also involve the well known, still insurmountable, deductive difficulties of the problem of the gravitative behaviour of three or more mutually attracting bodies), would result in the definition of physical quantities that, according to the theory, would have to be expressed, in general, by irrational numbers. But actual physical measurements can never even appear to verify any values but those expressed in rational numbers. Theory, in a word, demands, in such cases, an absolute precision in the definition of certain ideal entities. Measurement, in its empirical sense, never is otherwise than an approximation, and at best, when absolutely compared with the ideal, a rough one.

Why such concepts, which can never be shown to represent with exactness any physical fact, are nevertheless of such value for physical science, our methodological study has now shown us. *Their very unverifiability, as exactly defined concepts about the physical world, is the source of their fecundity as guides to approximate physical verification.* For what the observers verify are the detailed, even if but approximate correspondences between very large samples of empirical data, and samples of the consequences of hypotheses. The exactness of the theoretical concepts enables the consequences of hypotheses to be computed, that is, deductively predetermined, with a wealth and variety which far transcend precise physical verification, but which, for that very reason, constantly call for and anticipate larger and larger samples of facts of experience such as can furnish the relative and approximate verifications. *It is with theoretical science as it is with conduct. The more unattainable the ideals by which it is rationally guided, the more work can be done to bring what we so far possess or control into conformity with the ideal.*

The order-systems, viewed as ideals that our thought at once, in a sense "creates," and, in a sense "finds" as the facts or "entities" of a purely logical (and not of a physical) world, are therefore to be studied with a true understanding, only

when one considers them in abstraction from the "probable" and "approximate" exemplifications which they get in the physical world.

§ 16. Yet the logician also, in considering his order-types, is not abstracting from *all* experience. His world too is, in a perfectly genuine sense, empirical. We have intentionally used ambiguous language in speaking of his facts as *either* his "creations" *or* his "data." For if we say that, in one sense, he seems to "create" his order-types (just as Dedekind, for instance, calls the whole numbers "*freie Schöpfungen des menschlichen Geistes*"), his so-called "creation" is, in this case, *an experience of the way in which his own rational will, when he thinks, expresses itself.* His so-called "creation" of his order-types is in fact a finding of the forms that *characterize all orderly activity, just in so far as it is orderly*, and is therefore no capricious creation of his private and personal whim or desire. In his study of the Science of Order, the logician *experiences the fact that these forms are present in his logical world, and constitute it, just because they are, in fact, the forms of all rational activity.* This synthetic union of "creation" and "discovery" is, as we shall see, the central character of the world of the "Pure Forms."

A survey of the forms of order may therefore well begin by viewing them *empirically*, as a set of phenomena presented to the logician by the experience which the theoretical or deductive aspect of science furnishes to any one who considers what human thought has done. The most notable source of such an experience is of course furnished by the realm of the mathematical sciences, whose general business it is *to draw exact deductive conclusions from any set of sufficiently* precise hypotheses. If one considers the work of Mathematics,—analyzing that work as, for instance, the Italian school of Peano and his fellow workers have in recent years been doing,—one finds that the various Mathematical Sciences use certain fundamental concepts and order-systems, and that they depend for their results upon the properties of these concepts and order-systems. Let us next simply report, in an outline sketch, what some of these concepts and systems are.

§ 17. *Relations.* One "concept," one "logical entity," or (to use Mr. Bertrand Russell's term, employed in his *Principles of Mathematics*) one "logical constant," which is of the utmost

importance in the whole Theory of Order, is expressed by the term *Relation*. Without this concept we can make no advance in the subject. Yet there is no way of defining this term *relation* without using other terms that, in their turn, must presuppose for their definition a knowledge of what a relation is. In order, then, not endlessly to wait outside the gate of the Science of Order, for some "presuppositionless" concept that can show us the way in, we may well begin with some observations that can help us to grasp what is meant when we speak of a relation. A formal definition "without presuppositions" is impossible, whenever we deal with any terms that are of fundamental significance in philosophy.

Any object, physical or psychical or logical, whereof we can think at all, possesses *characters*, *traits*, *features*, whereby we distinguish it from other objects. Of these characters, some are *qualities*, such as we ordinarily express by *adjectives*. Examples are *hard*, *sweet*, *bitter*, etc. These qualities, as we usually conceive of them, often seem to belong to their object without explicit reference to other objects. At all events they may be so viewed. When we think of qualities, as such, we abstract from other things than the possessors of the qualities, and the qualities themselves. But, in contrast with *qualities*, the *relations* in which any object stands are *characters that are viewed as belonging to it when it is considered with explicit reference to, that is, as in ideal or real company with another object, or with several other objects.* To be viewed as a *father* is to be viewed with explicit reference to a child of whom one is father. To be an *equal* is to possess a character that belongs to an object only when it exists along with another object to which it is equal; and so on.

In brief, *a relation is a character that an object possesses as a member of a collection* (a pair, a triad, an *n*-ad, a club, a family, a nation, etc.), and which (as one may conceive), would *not* belong to that object, were it not such a member. One can extend this definition from any one object to any set of objects by saying that a relation is a character belonging to such a set when the members of the set are either taken together, or are considered along with the members of still other sets.

It is often assumed that relations are essentially *dyadic* in their nature; that is, are characters which belong to a member of a pair *as* such a member, or to the pair itself as a pair. The

relation of a *father*, or that of an *equal*, or that of a *pair of equals*, may be viewed as such a dyadic relation. But, as a fact, there are countless relations which are *triadic, tetradic,—polyadic*, in any possible way. When, for instance, is an object a *gift*? When, and only when there exists the triad: *giver, person or other entity whereto* something is *given*, and *object given*. When is an object a legal debt? Only, in general, when *creditor, debtor, debt*, and *consideration* or other *ground* for which or by virtue of which the debt has been incurred, exist. So that the *debtor*-relation: "a owes b, to c, for d," is in general a *tetradic* relation. Relations involving still more numerous related objects or terms are frequent throughout the exact sciences.

If a relation is dyadic, we can readily express the proposition which asserts this relation by using the symbol (a R b), meaning: "The entity a stands in the relation R to b." Whenever the proposition (a R b) is true, there is always also a relation, often symbolized by $\breve{R}$, in which b stands to a. This may be called the inverse relation of the relation R. Thus if: "a is father of b," "b is child of a;" and if one hereby means "child of a father" the relation *child of* is, in so far, the inverse of the relation *father of*.

If a relation is polyadic, then such symbols as R ($a\ b\ c\ d \ldots$), meaning "a, b, c, d, etc. (taken in a determinate order or way which indicates the place of each in the relational n-ad in question), stand in the (polyadic) relation R." Thus, with due definition of terms R ($a\ b\ c\ d$) may be used to symbolize the assertion: "a owes b to c for (or in consideration of) d;" and so on.

§ 18. *Logical Properties of Relations.* Relations are of such importance as they are for the theory of order, mainly because, in certain cases, they are subject to exact laws which permit of a wide range of deductive inference. To some of these laws attention must be at once directed. They enable us to classify relations according to various *logical properties. Upon such properties of relations all deductive science depends. The doctrine of the Norms of deductive reasoning is simply the doctrine of these relational properties when they are viewed as lawful characteristics of relations which can guide us in making inferences, and thus Logic as the "Normative Science" of deductive inference is merely an incidental part of the Theory of Order.*

Dyadic relations may be classified, first, as *Symmetrical* and

Non-symmetrical relations. A symmetrical dyadic relation is sometimes defined as one that is *identical with its own inverse relation.* Or again, if S is a symmetrical relation, then, whenever the assertion (a S b) is true, the assertion (b S a) is true, whatever objects a and b may be. The relation of *equality*, symbolized by $=$, is a relation of this nature, for if $(a = b)$, then always $(b = a)$.

If a relation is *non*-symmetrical, various possibilities are still open. Thus, if R be a non-symmetrical relation, and if (c R d), the relation R may be such that the assertion (d R c) is *always excluded* by the proposition (c R d), so that both cannot be true at once of whatever (c, d) one may use as the "terms" of the relation, then, in this case, the relation R is *totally* non-symmetrical. Russell proposes to call such relation Asymmetrical. The relation "greater than" is of this type in the world of quantities. But in other cases the relation R may be such that (c R d) does not exclude (d R c) in *every* instance, but only in certain instances. In the case of different relations, the exceptional instances may be for a given R, unique, or may be many, and may be in certain cases determined by precise subordinate laws of their own. Thus it may be the law that (c R d) excludes (d R c), unless some other relational proposition (e R$'f$) is true; while if (e R$'f$) is true, then (c R$'$ d) necessitates (d R c); and so on.

Without reference to the foregoing concept of symmetry, the dyadic relations may be classified afresh, by another and independent principle, which divides them into Transitive and Non-Transitive relations. This new division is based upon considerations which arise when we consider *various pairs* of objects with reference to some one relation R. If, in particular, (a R b) and (b R c), the relation R may be such that (a R c) is, under the supposed conditions *always* true, whatever the objects (a, b, c) may be, then in this case the relation R is *transitive.* If such a law does *not* universally hold, the relation R is non-transitive. The relation, *equal to*, is a transitive relation, according to all the various definitions of equality which are used in the different exact sciences. The so-called "axiom" that "Things equal to the same thing are equal to each other" is, in fact, a somewhat awkward expression of this transitivity, which, by definition, is always assigned, in any exact science to the relation $=$. The expression is awkward, because, by the use of "each other" in the so-called "axiom," the *transitivity* of

the relation $=$, is so stated as not to be clearly distinguished from the *symmetry* which also belongs to the same relation. Yet *transitivity and symmetry are mutually independent relational characters.* The relations, "*greater than*," "*superior to*," etc., are, like the relation $=$, transitive, but they are *totally non-symmetrical.* The relations "*opposed to*," and "*contradictory of*" are both of them *symmetrical*, but are also *non-transitive.*

Fewer formulations of this general type have done more to confuse untrained minds than the familiar "axiom:" "Things equal to the same thing are equal to each other," because the form of expression used suggests that the relation, $=$, *possesses* its transitivity *because of* its symmetry. Everybody easily feels the symmetry of the relation $=$. Everyone admits (although usually without knowing whether the matter is one of definition, or is one of some objectively necessary law of reality, true apart from our definitions), that the relation $=$ is transitive. The "axiom" suggests by its mode of expression that this symmetry and this transitivity are at least in this case, necessarily united. The result is a wide-spread impression that the symmetry of a relation always implies some sort of transitivity of this same relation,—an impression which has occasionally appeared in philosophical discussions. But nowhere is a sharp distinction between two characters more needed than when we are to conceive them as, in some special type of cases necessarily united, whether by arbitrary definition or by the nature of things.

If some dyadic relation, say X, is *non-transitive*, then there is at least one instance in which the propositions $(d\ X\ e)$ and $(e\ X\ f)$ are both of them true of some objects (d, e, f), while $(d\ X\ f)$ is false. As in the case of the non-symmetrical relations, so in the case of the non-transitive relations, this *non-transitivity*, like the before mentioned *non-symmetry*, may appear in the form of an universal law, forbidding for a given relation R *all* transitivity; or else in the form of one or more special cases where a given relation does not conform to the law that the principle of transitivity would require. These special cases may be themselves subject to special laws. A relation, T, is *totally non-transitive*, in case the two assertions $(a\ T\ b)$ and $(b\ \mathrm{T}\ c)$ if both at once true, exclude the possibility that (a T c) is true. Thus if "a is father of b" and "b is father of c," it is impossible that "a is father of c" should be true. The relation

father of, is both totally non-symmetrical and also totally non-transitive. That relation between propositions which is expressed by the verb "contradicts," or by the expression "is contradictory of," is symmetrical, but totally non-transitive. For propositions which contradict the same proposition are mutually equivalent propositions. The relation "*greater than*," as we have seen, is transitive, but totally non-symmetrical. The relation = is both transitive and symmetrical. And thus the mutual independence of transitivity and symmetry, as relational properties, becomes sufficiently obvious.

Still a third, and again an independent classification of dyadic relations appears, when we consider the *number of* objects to which one of two related terms can stand, or does stand, in a given relation R, or in the inverse relation $\breve{R}$. If "a is father of b," it is possible and frequent that there should be several other beings, *c*, *d*, etc., to whom *a* is also father. If "*m* is twin-brother of *n*," then, by the very definition of the relation, there is but *one* being, viz. *n*, to whom *m* can stand in this relation. If "*e* is child of *f*," there are *two* beings, namely the father and the mother, to whom *e* stands in this relation. In a case where the estate of an insolvent debtor is to be settled, and where the debtor is a single person (not a partnership nor yet a corporation), then the transactions to be considered in this one settlement may involve many creditors, but, by hypothesis, only one debtor, so far as this insolvent's estate alone is in question. Here, there are then several beings, (p, q, r, etc.), of each of whom the assertion can be made:—"*p* is creditor of *x*." But so far as this one case of insolvency alone is concerned, all the creditors in question are viewed as a *many* to whom only *one* debtor corresponds, as *the* debtor here in question.

The questions suggested by such cases are obviously capable of very variously multiplex answers, according to the relational systems concerned. Of most importance are the instances where some general law characterizes a given relation *R*, in such wise that such questions as the foregoing cases raise can be answered in universal terms. The principal forms which such laws can take are sufficiently indicated by the three following classes of cases:—

1. The relation *R* may be such that, if (*a* R *b*) is true of some pair of individual objects (a, b), then, in case we consider

one of these objects, *b*, there are or are possible *other* objects, besides *a*,—objects *m*, *n*, etc.,—of which the assertions (m R b), (n R b), etc., are true; while at the same time, if we fix our attention upon the other member of the pair, *a*, there are other objects (*p*, *q*, *r*) either actual, or, from the nature of the relation *R*, possible, such that (a R p), (a R q), etc., are true propositions. Such a relation *R* is called by Russell and others a "*many-many*" relation. The laws that make it such may be more or less exact, general and important. Thus the relation "*1° of latitude south of*" is such a "many-many" relation, subject to exact general laws.

2. The relation *R* may be such that, when (a R b) is true of some pair (a, b), the selection of *a* is uniquely determined by the selection of *b* while, given *a*, then, in place of *b*, any one of some more or less precisely determined set of objects could be placed. Thus if "*a* is sovereign of *b*," where the pair (*a*, *b*) is a pair of persons, and where the relation *sovereign of* is that of some one wholly independent kingdom (whose king's sovereign rights are untramelled by feudal or federal or imperial relationships to other sovereigns),—then, by law, there is only *one a* whereof the assertion: "*a* is sovereign of *b*" is true. But if we first choose *a*, there will be many beings that could be chosen in place of *b*, without altering the truth of the assertion. A case of such a relation in the exact sciences is the case "*a* is centre of the circle *b*." Here, given the circle *b*, its centre is uniquely determined. But any one point may be the centre of any one of an infinite number of circles. Such a relation *R* is called a "one-many" relation. Its inverse $\breve{R}$ would be called a "many-one" relation.

3. A relation *R* may be such that (whether or no there are many different pairs that exemplify it), in case (*a R b*) is true of any pair whatever, the selection of *a* uniquely determines what *one b* it is of which (*a R b*) is true, while the selection of *b* uniquely determines what *a* it is of which (*a R b*) is true. Such a relation is called a "one-one" relation. Couturat prefers the name "bi-univocal" relation in this case. The "one-one" relations, or, as they are often called "one-one correspondences," are of inestimable value in the order systems of the exact sciences. They make possible extremely important deductive inferences, for example those upon which a great part of the modern "Theory of Assemblages" depends.

The various classifications of dyadic relationships that have now been defined, may be applied, with suitable modifications, to triadic, tetradic, and other polyadic relations. Only, as the sets of related terms are increased, the possible classifications become, in general, more varied and complicated. A few remarks must here suffice to indicate the way in which such classifications of the polyadic relations would be possible.

If the symbol *S* (*a b c d*...) means: "The objects *a*, *b*, *c*, *d*, etc., stand in the symmetrical polyadic relation *S*," then the objects in question can be mutually substituted one for another, *i.e.* the symbols *a*, *b*, *c*, etc., can be interchanged in the foregoing expression, without altering the relation that is in question, and without affecting the truth of the assertion in question. This is for instance the case if S (*a b c d*...) means: "*a*, *b*, *c*, *d*, ... are fellow-members of a certain club," or: "are points on the same straight line," so long as no *other* relation of the "fellow-members" or of the "points" is in question except the one thus asserted. In such cases *S* (*a b c d*...), *S* (*b c d a* ...), etc., are equivalent propositions. Such a relation *S* is polyadic and symmetrical. The relation R, expressed by the symbol R (*a b c d*), is non-symmetrical (partially or totally) if in one, in many, or in all cases where this relation is thus asserted there is some interchanging of the terms or of the objects,—some substituting of one for another,—which is not permitted without an alteration of the relation R, or a possible destruction of the truth of the relational proposition first asserted. This is the case if R (*a b c d*) means: "*a* owes *b* to *c* for, or in consideration of *d*;" or, in a special case "*a* owes *ten dollars* to *c* for *one week's wages*." Such a relation is non-symmetrical. The number of terms used greatly increases the range of possibilities regarding what sorts of non-symmetry are each time in question; since, in some cases, *certain* of the terms of a given polyadic relational assertion can be interchanged, while others cannot be interchanged without an alteration of meaning or the change of a true into a false assertion. Thus if the assertion *R* (*a b c d*) means "*a* and *b* are points lying on a certain segment of a straight line whose extremities are *c* and *d*,"—then *a* and *b* can be interchanged, and *c* and *d* can be interchanged, without altering the truth or falsity of the assertion; but if the pair (*a*, *b*) is substituted for the pair (*c*, *d*), and conversely, the assertion would *in general* be changed in its meaning,

and might be true in one form, but false when the interchange was made. Consequently we have to say, in general, that a given polyadic relation, R, is symmetrical or non-symmetrical *with reference* to this or that pair or triad or other partial set of its terms, or with reference to this or that *pair of pairs*, or *pair of triads*, of its terms; and so on. In case of complicated order-systems, such as those of *functions* in various branches of mathematics, or of *sets of points*, *of lines*, etc., in geometry, the resulting complications may be at once extremely exact and definable, and very elaborate, and may permit most notable systems of deductive inferences.

In place of the more elementary concept of transitivity, a more general, but at the same time more plastic concept, in terms of which certain properties of polyadic relations can be defined, is suggested by the process of *elimination*, so familiar in the deductive inferences of the mathematical sciences. Suppose R $(a\ b\ c\ d)$ is a tetradic relation, symmetrical or non-symmetrical; suppose that the relation is such that if the propositions R $(a\ b\ c\ d)$ and R $(c\ d\ e\ f)$ are at once true, then R $(a\ b\ e\ f)$ necessarily follows. A very notable instance of such a relation exists in the case of the "entities of Pure Logic" of which we shall speak later. We could here easily generalize the concept of transitivity so as to say that this relation R is "transitive by pairs." But such transitivity, as well as the transitivity of a dyadic relation, is a special instance of a general relational property which *permits the elimination of certain terms that are common to two or more relational propositions, in such wise that a determinate relational proposition concerning the remaining terms can be asserted to be true in case* the propositions with which we began are true. Let the symbol α represent, not necessarily a single object, but any determinate pair, triad, or n-ad of objects. Let β represent another such determinate set of objects, and γ a third set. Let R and R' be polyadic relations such that R $(\alpha\ \beta)$ and R$'$ $(\beta\ \gamma)$. The first of these symbols means the assertion: "The set of objects consisting of the combination of the sets α and β (taken in some determinate mode or sequence), is a set of objects standing in the relation R." The second symbol, viz. R' $(\beta\ \gamma)$ is to be interpreted in an analogous way. Hereupon, suppose that either always, or in some definable set of cases, the propositions R $(\alpha\ \beta)$ and R' $(\beta\ \gamma)$, if true together, imply that R$''$ $(\alpha\ \gamma)$, where

R″ is some third polyadic relation, which may be, upon occasion, identical with either or both of the foregoing relations, *R′* and *R*. In such a case, the information expressed in *R* (α β) and R′ (β γ) is *such as to permit the elimination of the set or collection* β, so that a determinate relational proposition, results from this elimination. It is plain that transitivity, as above defined, is a special instance where such an elimination is possible.[1]

With regard to the "one-one," "many-one" and "many-many" classification of dyadic relations, we may here finally point out that a vast range for generalizations and variations of the concepts in question is presented, in case of triadic, and, in general, of polyadic relations, by the "operations" of the exact sciences,—operations which have their numerous more or less "approximate" analogues in the realm of ordinary experience. These operations make possible deductive inferences whose range of application is inexhaustible.

An "operation," such as "addition" or "multiplication," is (in the most familiar cases that are used in the exact sciences) founded upon a *triadic* relation. If *R* (*a b c*) means "The sum of *a* and *b* is *c*," or in the usual symbolic form, $a + b = c$, then the triadic relation in question is that of two numbers or quantities to a *third* number or quantity called their "sum." As is well known, the choice of two of these elements, namely the choice of the *a* and *b* that are to be added together (the "summands"), determines *c* uniquely, in ordinary addition. That is, to the pair (*a*, *b*) the third element of the triad (*a*, *b*, *c*) *uniquely corresponds*, in case R (*a b c*) is to be true. On the other hand, given *c*, the "sum," there are in general, various, often infinitely numerous, pairs (*d*, *e*), (*f*, *g*,), etc., of which the propositions, $d + e = c$, $f + g = c$, etc., may be true. But in case of ordinary addition if *c*, the "sum," is first given, and if then *one* of the "summands," say *a*, is given, the other, say *b*, can always be

[1] In the closing chapter of his *Psychology*, in a beautiful sketch of the psychological aspects of scientific thinking, Professor Wm. James characterizes the transitivity of those dyadic relations, which are so often used in the natural sciences, by saying that the objects whose relations are of this transitive type follow what he calls "The axiom of skipped intermediaries." This is a characteristically concrete way of stating the fact that *one* main deductive use of transitivity, as a relational property, lies in the fact that it permits certain familiar *eliminations*. If, namely: "*a* is greater than *b*, and *b* is greater than *c*," *we may eliminate the intermediary b*, and conclude deductively that *a* is greater than *c*. We are here concerned, in our text, with the fact that dyadic transitivity is only a special instance of the conditions that make elimination in general possible, and that determine a whole class of Norms of deductive inference.

found (if the use of "negative" numbers or quantities is indeed permitted in the system with which we are dealing), and, when found, is then uniquely determined. Triadic relations, such as that which characterizes addition, may therefore be subject to precise laws whereby, to one element, or to two elements which are to enter into a triad, either one or many ways of completing the triad may correspond, these possible ways varying with the relational proposition whose truth is to be, in a given case, asserted or denied, or is to remain unchanged through the substitution of various new objects for those already present in a given triad.

The "operations" of the exact sciences are of inestimable importance for all the order-systems in terms of which precise theories are defined and facts are described. It is not necessary that they should precisely resemble, in their relational properties, either the "multiplication" or the "addition" of the ordinary numbers and quantities. A glance at their possible varieties (as these are discussed in connection with modern "group-theory," or as a part of the treatment of the various "algebras" which newer mathematics has frequent occasion to develop), will readily show to any thoughtful observer the absurdity of the popular opinion, still often entertained by certain philosophical students, that "mathematics is the science of quantity." The "quantities" are objects that are indeed vastly important. Their "order-system" is definable in terms of a few important properties of certain dyadic and triadic relations. *All our power to reason deductively about quantity depends upon these few relational properties*, whose consequences are nevertheless inexhaustibly wealthy. But the algebra of quantity is *one* only of infinitely numerous algebras whose operations are definable in terms of triadic relations. And there is no reason why other operations should not be defined in terms of tetradic, and, in fact of n-adic relations. The "Algebra of Pure Logic" is, in fact, as Mr. Kempe has shown, the symbolism of a system whose "operations" are superficially viewed, triadic, but are really founded upon tetradic relations (see § 24, below). And mathematical science includes within its scope the deductive reasonings possible in case of all these order-systems, and capable of being symbolized by all these algebras.

§ 19. *Classes.* In describing relations and their properties, we have inevitably presupposed the familiar concept of

a *set* or *collection*, *i.e.* of a *class* of objects as already known. *Relations are impossible unless there are also classes.* Yet if we attempt to define this latter concept, we can do so only by presupposing the conception of *Relation* as one already understood. As we have already pointed out, such a "circle in definition" is inevitable in dealing with all philosophical concepts of a fundamental nature.

The concept of a Class or Set or Collection or Assemblage (*Menge*) of objects, is at once one of the most elementary and one of the most complex and difficult of human constructions. The apparent commonplaces of the Socratic-Platonic Methodology, and their intimate relation to the profound problems of the Platonic Metaphysic, which we touched upon in § 3, have shown us from the outset how the most obvious and the deepest considerations are united in this problem. The "burning questions" of the new "Theory of Assemblages" as they appear in the latest logical-mathematical investigations of our days, illustrate surprisingly novel aspects of the same ancient topic.

The concept of a Class, in the logical sense, depends (1) Upon the concept of an *Object*, or *Element* or *Individual*, which *does or does not belong to a given class*; (2) Upon the concept of the *relation* of *belonging to*, i.e. *being a member of* a class, or of *not so belonging*; (3) Upon the concept of *assertions*, true or false, which *declare that* an object is or is not a *member of* a given class; (4) Upon the concept of a Principle, Norm, or Universal which enables us to decide which of these assertions are true and which are false.

The *first* of these concepts is in many ways the most problematic of all the concepts used in the exact sciences. What constitutes an Individual, what is the "principle of individuation," how are individuals known to exist at all, how are they related to universal types, how they can be identified in our investigations, or how they can be distinguished from one another, whether they can be "numerically distinct" and yet wholly or partially similar or identical,—these are central problems of philosophy, which we in vain endeavour to escape by asserting in the usual way that "individuals are presented to us as empirical objects, by our senses." Whoever has had occasion to study any problem involving the doubtful or disputed *identity* of any individual object, knows that *no* direct sense-experience

ever merely presents to us an individual object such as we conceive of, where we subject our processes of identification to exact rules and tests.

For *logical* purposes, an Individual Object is one that we *propose to regard at once as recognizable* or *identifiable throughout some process of investigation, and as unique within the range of that investigation, so that no other instance of any mere kind of object suggested by experience, can take the precise place of any one individual*, when we view ourselves as having found any individual object. Thus *to propose to treat* an object as always *recognizable* under certain conditions, and as such that no *substitute* for it is possible, in so far as we treat it as *this* individual,—all this involves an *attitude of will* which our sense-experience can illustrate and more or less sustain, but *can never prove to be necessary*, or present to us as successfully and finally warranted by mere data.

The concept of an individual is thus one whose origin and meaning are due to our will, to our interest, to so-called pragmatic motives. We actively *postulate* individuals and individuality. We do not merely find them. Yet this does not mean that the motives which guide our will in this postulate are wholly arbitrary, or are of *merely* relative value. *There are some active and voluntary attitudes towards our experience which we cannot refuse to take without depriving ourselves of the power to conceive any order whatever as present in our world.* Without objects conceived as unique individuals, we can have *no Classes.* Without classes we can, as we have seen, define *no Relations*, without relations we can have *no Order. But to be reasonable is to conceive of order-systems, real or ideal. Therefore, we have an absolute logical need to conceive of individual objects as the elements of our ideal order systems.* This postulate is the condition of defining clearly any theoretical conception whatever. The further metaphysical aspects of the concept of an individual we may here ignore. *To conceive of individual objects is a necessary presupposition of all orderly activity.*

An individual once postulated as present may be *classed with* other individuals. If the various individuals in question are *viewed as if they were* already given, the act of *classing them thus*, that is of asserting that these individuals belong in the same class, is again an act of will. Its value is so far *pragmatic.* We accomplish in this way some *purpose* of our own,

some purpose of treating things as for some special reason distinguished or, on the other hand, undistinguished. In this sense, *all classes are subjectively distinguished from other classes by the voluntarily selected Norms, or principles of classification which we use. Apart from some classifying will, our world contains no classes. Yet without classifications we can carry on no process of rational activity, can define no orderly realm whatever, real or ideal. In this sense, the act of defining at least some norms or principles of classification is an act whose logical value is not only pragmatic, but also absolute.* For a world that we might conceive as wholly *without* classes, would be simply no world at all. We could do nothing with it or in it. For to act, consciously and voluntarily, in any way whatever is to classify individuals into the objects that *do* and into those that *do not* concern, meet, serve, correspond to, stimulate or result from each sort of activity. Thus classes are in one sense "creations," in another sense absolute presuppositions of all our voluntary activity, and so of all our theories.

If we have in mind some norm of principle of classification, this norm inevitably defines at least one pair of classes, namely a given class and its *negative* or *contradictory* class. For if the class x is defined by a given norm, then the same norm defines the class consisting of whatever objects are not x, a class here to be symbolized by $\bar{x}$.

Whenever we set out to classify any region of our world, real or ideal, we of course always do so because we know, or at least postulate, that there are *some* individuals in that region to be classified. And considered with reference to a given norm, which defines a class x, these individuals will belong *either to* x *or else to* $\bar{x}$. But of course our norm does not of itself tell us whether there are any individuals, in the region to be classified, which *are* of the class x. We can, then, define a norm for a class x, and later discover that "Everything is $\bar{x}$," so that "There are no x's." In general, then, when we define by its norm the class x, either one of two assertions may turn out to be true about x. Either (1) "x has *no* member," or (2) "x has at least one member." Of these two assertions *one* is true, the other false, when uttered about any determinate class x. That is, these assertions are *mutually contradictory.*

A very vast range of the assertions of the exact sciences can be said to be of one or the other of these two comparatively

simple types. A class that has *no* members, a "nothing-class," an "empty class," or "zero-class" may be symbolized by 0. It is in that case a class sharply defined by its norm, but known *not* to contain any of the objects that we have chosen to regard or to define as the individuals of the world (real or ideal) with which we are dealing. If a class x has *no* members, its negative, viz. $\bar{x}$, comprises *everything* that belongs to the realm or (in the phrase of the English logician, De Morgan) to the "universe of discourse" with which we are dealing. The class *everything* can be symbolized by 1. Regarding 0 and 1 as classes, and using $=$ as the symbol, in the present case, of the relation of logical *equivalence* or *identity* between any two classes, we can assert, as formally true of any world, which for any reason, we can classify, that:—

$$(1)\ 0=\bar{1};\quad (2)\ \bar{0}=1.$$

That is, the class *nothing* and the class *everything* are negatives each of the other, whenever these terms are used of any one "universe of discourse" into which a definite classification has been introduced.

Given any two distinct classes, x and y, defined by different norms or principles of classification, then inevitably, and without regard to whether x and y are, either or both of them "zero," that is "empty" classes, the very definitions of x and of y require that two new *resulting* classes should be present, as classes that may or may not have members, in our classified world. These new classes are: (1) The "Logical Product" of the classes x and y, that is, the class of those objects in our "universe of discourse" that conform *at once* to the norm of x and to the norm of y, and that, therefore, belong at once to both the classes x and y; (2) The Logical Sum of the classes x and y, that is, the class of those objects that conform *either* to the norm of x *or* to the norm of y, and that therefore belong to *one at least* of the two classes (x, y). We symbolize by xy the logical product of x and y, and by $x+y$, their logical sum. In every extended discussion of classes logical sums and products are sure to occur.

Between two classes, p and q, there may or may not exist a certain relation which is of fundamental importance for all study of classes, and so for all exact science. This is the relation of *subsumption*. It is a relation non-symmetrical, but *not*

totally non-symmetrical. We may symbolize this relation by $-\!\!<$. If $p -\!\!< q$, then whatever conforms to the norm of p conforms to the norm of q; or, as we also may say, the class p is *included in* the class q. If $(p -\!\!< q)$ and $(q -\!\!< p)$ are at once true, then $(p = q)$. In case the relation $(p -\!\!< q)$ holds true, the logical product of p and $\bar{q}$ has *no* members, or in symbols, $p\bar{q} = 0$. The *subsumption* relation is transitive, that is :—

"If $(p -\!\!< q)$ and $(q -\!\!< r)$ then $(p -\!\!< r)$."

As the modern study of the topic has shown, *the entire traditional "theory of the syllogism" can be expressed as a sort of comment upon, and relatively simple application of, this transitivity of the subsumption-relation.* Thus does the theory of the "norms of thought" form merely a subordinate part of the theory of Logical Order.

One relation remains here to be explicitly characterized,—a relation often confounded with the subsumption relation, but carefully distinguished therefrom, in recent times by Frege, Peano, and Russell. It is *the relation in which an individual stands* to the class to which it belongs, and of which it is a member. The school of Peano symbolize this relation by ϵ. Thus, supposing i to be the name of an individual object, the symbol $(i \,\epsilon\, x)$ means: "i is a member of, that is, belongs to the class x." Since a class itself can be and sometimes is treated logically as an individual, in case this class is taken *as* one member of a *set of classes* (as, for instance, when one says: "The powers of 2, such as 2^2, 2^3, etc., form a class that is one of the classes of whole numbers"), we can suppose the proposition $x \,\epsilon\, y$ to be true of some class x, y being a class of classes. But in such a case, if $(i \,\epsilon\, x)$ and $(x \,\epsilon\, y)$, then the assertion $(i \,\epsilon\, y)$ is, in general, false. So that the ϵ-relation is *non-transitive*, while the relation $-\!\!<$, the subsumption relation is transitive. They are, then, quite different relations.

Any class, x, consists of the individuals, i, i', i'' . . ., whereof the corresponding assertions $(i \,\epsilon\, x)$, $(i' \,\epsilon\, x)$, etc., are true. From the formal point of view it is thus possible, and in fact, for certain logical purposes necessary, to develop the "Theory of Classes" upon the basis of the "Theory of Propositions." Propositions, themselves, have certain characteristic logical relations, of *contradiction*, *implication*, and so on. To these relations of propositions those relations of classes which we have named,

viz. *negation*, *subsumption*, etc., correspond in certain exact ways. There is therefore possible a "calculus of classes;" although the two doctrines have certain notable differences regarding the principles available for deductive purposes in each of them.

The assertions of the type ($i \epsilon x$) upon which classifications may be said to rest, have the aforesaid paradoxical character. They are, namely, the expressions of *postulates*, or voluntary acts, since all classification involves a more or less arbitrary norm or principle of classification. Yet the *laws* to which such propositions, as well as any logical system of classes are subject, are nevertheless exact, are definable (as we have seen) in terms of precise dyadic, triadic and tetradic relations, and *are not in the least arbitrary*. In fact, despite the arbitrariness of each individual classification, the general laws of logic possess an absoluteness which cannot conceivably be surpassed, and lie at the basis of all order-system and of all theory.

The only possible answer to the question as to *how* the absoluteness of the logical principles is thus consistent with the arbitrariness of each of the classifications which we make, lies in saying that the logical principles define precisely the nature of the "Will to act in an orderly fashion" or in other words of the "Will to be rational."

§ 20. *The Types of Order.* The foregoing concepts of Relation, of Relational Properties, and of Classes, have enabled modern mathematicians, and other students of logic, to define in exact terms a surprisingly vast range of order-systems. With almost dramatic suddenness the considerations which may have seemed so varied, disunited and abstract in the foregoing sketch, suddenly give us, when they are once properly combined, an insight into precisely what is most momentous about the order present in the worlds of Number, of Quantity, of Geometry, and of Theoretical Natural Science generally.

For, in the first place, what order-type is universally present wherever there is *any* order in the world? The answer is, *Serial Order*. What is a *Series*? Any row, array, rank, order of precedence, numerical or quantitative set of values, any straight line, any geometrical figure employing straight lines, yes all space, all time,—any such object involves serial order. Serial order may exist in two principal types, the "*open*" series, and the "*closed*" series or *cycle*. Since the latter type of order may

be reduced to the former by certain well-known devices, it suffices here to characterize any serial order that is "open," *i.e.* that does not return into itself. So viewed, a *Series* is :— *A class of individuals or elements such that there exists a single relation R, dyadic, transitive, and totally non-symmetrical, while this relation R is of such a nature that, whatever pair (a, b) of distinct elements of the class in question be chosen, either (a R b) or else (b R a) is true.* Since the relation *R* is by definition totally non-symmetrical, (*a R b*) and (*b R a*) cannot be true at once of any chosen pair of objects belonging to the series defined in terms of *R*. If we begin with any pair, (*c*, *h*), of elements of a given series, the "place" of any other element *a* or *g* is determined with reference to *c* and *h* by such assertions as (*a R c*) and (*c R g*), (*g R h*), etc., while the transitivity of the relation *R* *enables us to use such assertions as a basis for deductive inferences whenever two pairs with a common element appear in the course of our determinations.* Chains of inference, eliminations, etc., result. Thus, once more, certain norms of deductive inference are determined by relational properties.

Now in terms of the variations which this definition of *series* permits to be present in the classes and sub-classes of which a series may consist, an infinite variety of distinct serial types can be defined upon the basis of the single definition just stated, and of the logical properties of classes.[1]

The series of the positive whole numbers, for instance, is characterized by the fact that there is *one* member of the class in question, namely the *first*, which stands in a relation *R* to every other whole number, *R* being the transitive and totally non-symmetrical relation of "predecessor," while no positive whole number stands in the relation *R* to this first one; and by the further fact that whatever number (say 2, or *n*) one chooses, there is *one* number (say 3, or $n+1$) and *only one*, such that, while (n R $n+1$) is true, *no* whole number m exists such that (n R m) while (m R ($n+1$)). In this case ($n+1$) is called the *next successor of n*. And thus the relation "next successor"

[1] The use of the foregoing definition, and the classifications of possible serial types which the definition permits, have now become common property. The significance of the definition, and the wealth of ordinal properties that could be stated in terms of it, were gradually brought to light in the latter half of the nineteenth century through the researches of C. S. Peirce, of Dedekind, of Cantor, and of various other logical and mathematical writers. The results have been summed up, and placed in various new lights, in Russell's *Principles of Mathematics*.

is defined in terms of R, and of the absence of intermediates. A further characteristic of the whole numbers is this, that if any property Q belongs to the *first* whole number, and if Q is such that, in case Q belongs to any whole number, n, Q belongs to the "next successor" of n (say to $n + 1$), then Q belongs to *all* of the whole numbers. This characteristic of the whole number series is defined and applied by combining the other relational properties of the series with the logical properties of classes, and *is of the most fundamental importance for deduction throughout mathematical theory*. Thus still another norm of deductive reasoning is established for a certain class of cases.

Such simple considerations concerning classes and relations define then, the series of the whole numbers, and predetermine the inexhaustible wealth of the Theory of Whole Numbers. An extension of such an ordinal series "backwards" gives us the *negative whole numbers*. The series of the "*rational numbers*" can be characterized as to its ordinal type by defining the relation R, for that series, and also by choosing the elements of the series, *so* that whatever pair (i, k) of distinct rational numbers exists, such that $(i\ R\ k)$ is true, there also exists j different from i and from k such that $(i\ R\ j)$ and $(j\ R\ k)$. A series of this type is now called "*dense*." Upon the basis of the *dense* series of the rational numbers we can define another series, that of the "cuts" or *Schnitte* of the rational numbers. This new series is (in Dedekind's sense) "continuous." It is defined in terms of still another union of a certain sort of classification with the relational properties already in question. This series of the "cuts" of the rational numbers is the series of the "real numbers." And Cantor has worked out a more precise characterization of the properties of the continuous series of "real numbers" (the so-called "arithmetical continuum") by a still further synthesis of the properties of certain sub-classes which such a series contains, with the general properties of the relation R, whereby the series as a whole is determined.

In consequence, mathematical science is now in possession of a complete definition of the "arithmetical continuum" in purely ordinal terms.

But the numbers are not merely subject to *dyadic* ordinal relations. As usually employed in arithmetic and algebra, they are also subject to *triadic* relations, in terms of which the *operations* of ordinary addition $(a + b = c)$, and of multiplication

(pq = r) are defined. The momentous problem arises as to *how these triadic relations are themselves related to the dyadic ordinal relations of the number-series.* This problem has been attacked with complete success by the modern students of the foundations of mathematics. It has been shown, first, that the simple series of the *whole numbers*, defined as above, is such as to enable us to define *for that series* the operations of the addition and multiplication of its own terms upon the basis of considerations that involve solely the dyadic relational properties of this whole-number series as it stands. That is, in case of a series such as the whole numbers, positive and negative, the triadic relations involved in addition and in multiplication, can be defined in terms of the dyadic relations whereby the series is ordered. But in case of the dense series of the rational numbers, and still more in case of the "arithmetical continuum" of the "real numbers," and again yet more in case of the "complex numbers" of algebra, *such a reduction of the triadic relations of these numbers to the dyadic relations of the whole numbers can be accomplished only indirectly*, by means of special definitions, which enable us to regard these other series and in fact the whole system of the "complex numbers," as derived, through a sort of "logical genesis," from the original whole number series, by a *series of combinations* of the terms, classes, and relations, of the latter series, and by further combinations of the results of these first combinations. All this "genesis" we have not here room to follow. It is enough to say that the result of this research is to show that all the properties which make the numbers of ordinary algebra subject to the endlessly varied operations of *calculation*, can be reduced to properties which depend: (1) Upon the dyadic relations of order which hold in the whole-number system itself, and (2) Upon the properties and ordinal relations of certain derived logical entities (*pairs* of whole numbers, *classes* of these pairs, *pairs* of real numbers, etc.). And in brief, we can say: All the properties of the numbers which are used in ordinary algebra, are properties of their order-system, while this order-system is *indirectly definable* on the basis of the properties of the whole-number system, and of the properties of certain classes and relations of objects which the whole-number system enables us to define.

The number-system of ordinary algebra being once defined, it is possible to deal, in a systematic way, with the problems

which are presented by the physical and ideal *Quantities* with which mathematical theories so frequently deal. *Quantities* are objects, either physical or ideal, that fall into series by virtue of relations of the nature of *greater* and *less*. They have therefore their serial order-systems. They also, in general, are subject to relations of *equality*. In case they are *Intensive* Quantities, their order-systems are definable *only* by means of such dyadic relations, that is, by means of relations of *greater-less*, and by means of the symmetrical relation of *equality*. Extensive Quantities are such as, *over and above* these dyadic relations, of *greater*, *less*, *equal*, are subject to *triadic* relations in terms of which the *sum* of any two extensive quantities that belong to the same system of quantities can be defined. In the realm of the quantities, however, there is no *general* mode of "logical genesis" which makes it possible for us to define triadic relations of the type $(a + b = c)$ upon the sole basis of the dyadic relations *greater*, *less*, and *equal*. Herein the quantities differ, as logical objects, from the number-series viewed as pure algebra views the latter. The "logical genesis" of the rational and of the "real" numbers,—a genesis of which we have just made mention, has no precise and general correspondent process in the world of quantities. Therefore, those triadic relations of most sets of extensive quantities upon which their addition depends, are defined, either (1) upon the basis of empirical inductions (as is the case with physical weights, with quantities of energy, etc.), or (2) upon the basis of voluntarily assumed postulates (as is the case with many systems of ideal quantities, such as for instance the extensive quantities of Pure Metrical Geometry as they are usually treated), or (3) upon some union of postulate and of physical experience (as is frequently the case in the applications of geometry, and in such a science as Mechanics).[1]

Given, however, some workable and sufficiently general definition of a triadic relation upon which an addition-operation can be founded, then the number-system can be at once introduced into the theory of any system of quantities. The exactness of a physical theory of such a *set* of quantities depends

[1] In the very notable case of geometrical theory, a special form of reduction of the "metrical" to the "ordinal" properties of space-forms also exists, whereby the bases of metrical geometry can be indirectly reduced to principles that are stateable wholly in projective, that is, in ordinal terms. This case is of vast importance for the logic of geometry, but cannot further be studied here.

upon such an introduction. The order-system of such a realm of extensive quantities becomes correspondent to the order-system either of a *part* of the numbers, or of the *entire* system of the real or of the complex numbers. Thus, what makes deductive inference in the realm of quantity possible depends solely upon the ordinal properties of this realm.

The application of the foregoing principles regarding serial order-types to the theory and description of more complicated order-systems, involves a set of processes to which we have now made frequent reference, namely: The *Correlation of Series. Upon such correlations the whole theory of Mathematical Functions depends*,—a theory which admits of infinitely numerous variations and applications, and which plays its part in every extended and exact theoretical science. The norms of deductive inference which are definable here are numerous and complex, but vastly important.

The *simplest type of correlation* is that which takes place when a relation of "one-one correspondence" can be established between the members of two series, or between the members of definable parts of such series. In other cases, a "one-many" relation can be established, whereby to every member (say p) of a given series S, there corresponds some determinate number, a pair (q, r) or a triad (q, r, s) of elements, chosen from some series S′, or else so that q belongs to S′, r to S″, etc.; while, given $(q, r, \text{etc.})$, p is uniquely determined. The possibilities thus suggested may be still further varied without any necessary sacrifice of exactness of definition. In very numerous instances, especially where the operations possible in case of numbers and quantities are in question, we may have a correspondence and correlation of series so established that, to each of a set of pairs (p, q), or of triads (p, q, r) etc. (whereof p shall be chosen from one series, q from another or from the same series, and so on), there corresponds some determinate element x, or some set of elements $(x, y, z, \text{etc.})$, while the element x (or the set x, y, etc.) can be defined as elements of some series or order-system that thus *results from* or *that is definable in terms of the "functional relation,"* whose laws lie at the basis of the correlation in question. In general, let α, β, and γ be, not now single individuals viewed merely as such, but *pairs*, *triads*, or other sets of objects. Let the elements whereof each of the sets α, β, and γ, consists, be all chosen in a determinate way from certain series of objects

already defined (number-series, points on lines, series of lines or of other geometrical figures, physical quantities, etc.). Suppose that some general law exists which one can state in the form: "If $R'(\alpha)$ and $R''(\beta)$ are both of them true, $R'''(\gamma)$ is true." Then such a law establishes a *functional* relation, or a *system of functional relations*, amongst the various series from which the elements of α, of β, and of γ, respectively, are chosen.

For instance, $R'(\alpha)$ may stand for some combination of quantities of different forms of physical energy (coal burned, water-power supplied, etc.). These forms of energy may be combined in the production of certain industrial products. Each of these quantities, in a special case, will then be a member of its own series (weight of coal, amount of water used at a certain "head," etc.). $R''(\beta)$ may be a combination of the costs of these various forms of energy, when the energy is obtained under certain conditions. And then, again, each of these elements of cost will have its place in its own series, determined by a price-list (price of coal per ton, of water per cubic metre, etc.). Hereupon, in ways determined by the mode of production, by the waste or the use of energy, etc., there may correspond to a given combination $R'(\alpha)$ and a combination $R''(\beta)$, a given set of costs of a set of industrial products, expressed by $R'''(\gamma)$. In such a case the costs of the products will appear as in "*functional relations*" to the sources of energy used, and to the costs of each of these sources. Wherever such a correlation of series, or of sets or systems of series appears, the result is an Order determined by the correlations.

As Klein long since pointed out, the various types of Geometrical Science, the different geometries (metrical, projective, etc.), may be classified in terms of the "invariants" (that is, of the unchanging laws of the results of correlation) to which the different geometrical "transformations" are subject. And the geometrical "transformations" (projections, systematic deformations, dualities, inversions, etc.) involve correlations of sets of series such that (with the foregoing definition of the symbols used), $R'(\alpha)$, and $R''(\beta)$, etc., imply, as their combined result $R'''(\gamma)$, in ways which the relational properties of the geometrical world enable the geometer to define. In general a mathematical "transformation" means a definition of one system of relations by means of a correspondence with other systems of relations and of related terms. Its "invariant" is a law or a

relational property or construction which is exemplified by each and by all of the correlated systems.

A very highly important condition of the orderly character of the systems within which such "functional relations," and such "transformations" are possible, is the existence of relational properties that admit of *eliminations*, of the type discussed in our general account of relations (§ 18, near the close). *What transitivity is in the definition of a single series, the more general relational properties which permit elimination are, in the definition of the complex geometrical and physical order-systems which admit of definite and lawfully repeated correlations and transformations.*

It remains to say a word as to the significance of the Symmetrical Relations in the constitution of all such order-types. If $a = b = c = d$, etc., the set of objects between any two of which such a dyadic symmetrical transitive relation as $=$ obtains, may be called a *Level*. On a topographical map, the lines that indicate levels, the "contour-lines," run through points any two of which stand for physical points, on the surface mapped, such that they are at *equal* heights above some "base-level" (usually above sea-level). Isothermal lines, isobars, parallels of latitude, and countless other symbols for levels, are conspicuous features of the diagrams that are used to depict the orderly structure of real or ideal objects. Yet the members of such a level are not ordered by means of their symmetrical transitive *levelling* relations. They are ordered, if at all, in terms of serial relations, or in terms of the foregoing correlations of systems of series. Yet levelling processes and relations are constantly used in the definition of order-systems. The topographical map, or the "weather map" illustrates this fact. And the vast range of usefulness which the Equation has in mathematics is one of the best known features of that science. Why are relations which by themselves do *not* order, so useful in the definition of types of order?

The answer to this question is three-fold :—

1. The symmetrical relations, and especially the symmetrical transitive relations, enable us to classify, and so *form the basis for all the most exactly definable classifications of the Science of Order.*

2. For this very reason, many of the most important series in the theoretical sciences are *Series of Levels*. Such, for instance, are the series of contour-lines, isobars, etc., on a map.

3. And again, for the same reason, many of the most important *laws* of an ordered world are defined in terms of levels. The "invariants" of a system of "transformations" establish, in general, such levels. That is, when two or more systems are correlated through a "transformation," the results of such correlation leave certain relations that belong to each system unchanged by the passage from one system to the other. Thus a level is established. For instance, the law of the Conservation of Energy is a law expressed by asserting that, between any two states A and B of a given "closed system" in the physical world there obtains a certain symmetrical transitive relation, namely the relation expressed by saying that: "The total energy present in the system in the state A is equal in quantity to the total energy present in the system in the state B." In other words, the total energy remains invariant through the transformation. Thus the statement of the "invariant" law of any system of correlations or of transformations always includes some elements that can be expressed by symmetrical transitive relations. All this is the result of the same *inseparable union of the concepts of Class and of Relation*,—a union which we have illustrated from the beginning of our sketch of Order.

It will be noted, as we now look back, that the various norms of deductive inference, in all the various cases here in question, depend upon the relational properties of the order-systems which are under consideration, and so, in the last analysis, upon the properties of single relations. Thus Formal Logic, as a "Normative science," is incidental to the application of the Theory of Order to this or that process of deductive inference.

SECTION III.

THE LOGICAL GENESIS OF THE TYPES OF ORDER.

§ 21. IN our first section, the study of methodology showed us the relation of all scientific procedure to the Theory of Order. In our second section we have portrayed, in a largely empirical fashion, the Types of Order which characterize the exact sciences. Two of the concepts absolutely essential to the Theory of Order, we have already treated, indeed, so as to show *why they are necessary.* These are the concepts of Relation and of Class. For not only are these concepts actually used in the definition of every type of order, but as we have seen, *their necessity depends upon the fact that without them no rational activity of any kind is possible.* We have consequently insisted that these concepts unite in a very characteristic way—"creation" and "discovery," an element of contingency and an element of absoluteness. That a particular physical or psychical *relation*, such as that of father and child, should be present in the world, is as empirical a fact as the existence of colors or tones. That there should be physical objects to *classify*, this again is a matter of experience. And furthermore, every classification of real or of ideal objects is determined in any special instance, by a norm or principle of classification which we voluntarily choose. And in so far classifications are arbitrary, and may be said to be "creations" or "constructions." Yet, whatever else the world contains, if it only contains a reasonable being who knows and intends his own acts, then this being is aware of a certain relation, the relation between performing and not performing any act which he considers in advance of action. And thus relations amongst acts are in such wise necessary facts, that whoever acts at all, or whoever, even in ideal, contemplates *possible courses of action*, must regard at

least some of these relations as present in the realm of his conceived *modes of action.* In a similar fashion, as we have seen, every sort of action determines a kind of classification of some world, physical or ideal. In so far, therefore, as the nature of relations and of classes in general determines the existence and the meaning of types of orderly activity, these types of orderly activity, and the order-systems which express their nature, are both empirical objects, "found" (since we experience their presence in our world); and are also *necessary* objects, because if we try to conceive that they are *not* there, our very conception involves modes of action, and hence restores these necessary relations and classes to the world from which we had tried to banish them. We "construct" relational systems and classes in our ideal world. But we also "find" that at least *some* of these constructions are necessary.

A frequently asserted modern view, to which we have made some reference in the foregoing, namely the view called Pragmatism, asserts that all truth, including logical truth, has its basis in the fact that our hypotheses, or other assertions, prove to be *successful*, or show by their empirical workings that they meet the needs which they were intended to meet. From this point of view the logical hypothesis: "That there are classes, relations, and order-systems," would be true merely in so far as the acts of conceiving such objects, and of treating them as real, have, under the empirical conditions under which we do our thinking, a successful result. And thus logical truth, and the logical existence and validity of classes, of relations, and of the various types of order, would stand in the same position in which all the "working hypotheses" of an empirical science stand. These order-systems would exist, and their laws would be valid, precisely in so far as such ways of actively conceiving of the world have successful workings.

But, in the foregoing, we have already indicated that, so far as the existence of classes and of relations in general is in question, and in so far as the validity of certain logical laws is concerned, we are obliged to maintain a position which we may characterize by the term Absolute Pragmatism. This position differs from that of the pragmatists now most in vogue. There are *some* truths that are known to us *not* by virtue of the special successes which this or that hypothesis obtains in particular instances, but by virtue of the fact that *there are certain modes of*

activity, certain laws of the rational will, which we reinstate and verify, through the very act of attempting to presuppose that these modes of activity do not exist, or that these laws are not valid. Thus, whoever says that there are no classes whatever in his world, inevitably classifies. Whoever asserts that for him there are no real relations, and that, in particular the logical relation between affirmation and denial does not exist, so that for him *yes* means the same as *no*,—on the one hand himself asserts and denies, and so makes a difference between *yes* and *no*; and, on the other hand, asserts the existence of a relational *sameness* even in denying the difference between *yes* and *no*.

In brief, whatever actions are such, whatever types of action are such, whatever results of activity, whatever conceptual constructions are such, that the very act of getting rid of them, or of thinking them away, logically implies their presence, are known to us indeed both empirically and pragmatically (since we note their presence and learn of them through action); but they are also absolute. And any account which succeeds in telling what they are has absolute truth. Such truth is a "construction" or "creation," for activity determines its nature. It is "found," for we observe it when we act.

It consequently follows that whoever attempts to justify the existence of any of the more complicated systems of order that we have been describing in the foregoing section, has a right to seek for some absolute criterion, whereby he may distinguish what systems of order are necessary facts in the world,—that is, in the world that the logician has a right to regard as necessary,—and what, if any, amongst these forms are either capricious, and unnecessary, or else are suggested by the particular facts of experience in such wise as to remain merely contingent.

The logician's world is the world of hypotheses, and of theories, and of the ideal constructions that are used in these theories and hypotheses. Now theories and hypotheses may be merely suggested to us by physical phenomena, so that, if we had different sensations from our present ones, or if our perceptions followed some other routine than the observed one, we should have no need for these hypotheses and resulting theories. In so far, the hypotheses are contingent, and the theories have only conditional value. Furthermore, some of our activities are indeed arbitrary, so that we may, as the common expression is,

"do as we like." And when such modes of activity play their part in the choice or in the definition of our hypotheses, the logician cannot regard them as necessary. But such logical facts as the difference between *yes* and *no*, are not dependent on the contingent aspect of our sensations, but on our rational consciousness of what we intend to do or not to do. Such facts have not the contingency of the empirical particulars of sense. And some modes of action, such as affirmation and denial, are absolute modes.

We can indeed suspend the process of affirmation and denial, but only by suppressing every rational consciousness about what we ourselves purpose to do. The particular deed may be arbitrary. But the absolute modes of activity just suggested are not arbitrary. We cannot choose to do without them, without seeking to choose, since choice is action, and involves, for instance, the aforesaid difference between affirming and denying that we mean to do thus and thus.

§ 22. Considerations of this sort show us that the Theory of Order must undertake a task which the foregoing sketch has only suggested. It now appears that the logician's world has in it *some* necessary elements and laws upon which order-systems may be founded. But this fact does not of itself suffice to tell us *what ones* amongst the enormously complicated order-systems of mathematics include contingent and arbitrary elements, and what ones are indeed in such wise necessary that whoever knows what his own orderly activity is, must recognize that these order-systems belong to his logical world. Let us illustrate the issue thus brought to our attention.

In the physical world, we meet with the difference between *greater* weights and *less* weights. We meet with this difference empirically, and test it by experiment. The result is that we get tests, such as the balance, whereby we can arrange physical weights in a series of Levels, each level consisting of observed weights any two of which are equal, while the series of these levels is determined by the transitive and totally non-symmetrical relation of greater and less. The familiar operations of putting two weights in one scale-pan of a balance and finding a single weight that, put into the other scale-pan, will balance them, enables us to define for the weights an operation of *summation*, —a *triadic relation* of weights. This operation empirically conforms to the laws of the addition of quantities. Hereupon,

by processes not further to be followed in this discussion, we establish an ideal and hypothetical correlation between physical weights and the number-system of arithmetic; and so the physical world, so far as weights are concerned, is conceived in orderly terms, in a way that makes many physical theories logically possible.

Now it is obvious that the existence of physical weights, and that all of the aforegoing relations, so far as they are physical relations, are, from our human point of view, both empirical and contingent. We can easily conceive of a physical world without any such phenomena. For if all our knowledge of nature came to us through sight and smell alone, in the form of colours, odours, etc., and if we never saw anything that suggested to us the comparing of weights, we should of course know of no physical facts that would define for us this order-system.

On the other hand, in defining the system of the weights as in the case of any other extensive quantities, we use our empirical facts for the sake of establishing some kind of correlation between the quantities of our physical world and the facts and laws of the number-system. *But what shall we say about the number-system itself?* It is a system, whose first principles can be stated as hypotheses of a very general nature concerning objects that can be distinguished, numbered, etc. Is our experience of the existence of such objects altogether as contingent as our experience of the existence of weights in the physical world? One obvious answer is suggested by the fact that we can apply the system of the whole numbers to characterize our own acts. Any orderly succession of deeds in which we pass from one to the next has certain of the characters of the series of ordinal whole numbers. In any orderly activity that we begin, we have a first act followed by a second, followed by a third, and so on. It therefore may occur to our minds that our knowledge of at least the whole numbers, like our knowledge of the difference between *yes* and *no*, may be founded upon the consciousness of our own activity and some of its necessary characters. But this view, when first stated, meets with the very obvious difficulty that, during our actual human lives, we perform only at best a very limited number of distinct acts, while the whole number-series, as the mathematician conceives it is an *infinite* sequence. Furthermore, nothing about the empirical nature of our activity as human beings seems to

determine the number of deeds that we shall do in our short lives. But the whole numbers of the mathematician present themselves as an order-system such that every member of the series *must* have its next successor. No mere observation of the contingent sequence of our own empirical deeds can therefore by itself warrant the necessity that the infinite sequence of the whole numbers should have a place in the logician's world at all.

Yet this consideration is, once more, only a suggestion of a difficulty, but not a decisive proof that the whole-number series is devoid of absolute necessity. For perhaps there is indeed something about the nature of our activity, in so far as it is rational,—something which necessitates a *possible* next deed after any deed that has been actually accomplished. And this possibility may prove to have something absolute about it. Such considerations deserve at least a further study.

To sum up:—The order-systems of mathematics are suggested in *some* cases by contingent empirical phenomena. In *other* aspects these order-systems may prove upon analysis to be absolutely necessary facts, in the same sense in which the existence of classes and relations of some sort are necessary facts in our world. And thus may be stated *the central problem of the Theory of Order.* This problem is:—*What are the necessary "logical entities," and what are their necessary laws? What objects must the logician's world contain? What order-systems must he conceive, not as contingent and arbitrary, but as so implied in the nature of our rational activity that the effort to remove them from our world would inevitably imply their reinstatement,* just as the effort to remove relations and classes from the world would involve recognizing both classes and relations as, in some new way, present.

It is precisely in this form that the problem of the theory of order appears to be, at the present time, undergoing a most progressive series of changes, enlargements, and enrichments. The "Deduction of the Categories" is taking on decidedly new forms in recent discussion. The principles that will enable us in the future to make an indubitable endless progress in this field at least possible, remain very briefly to be considered as our sketch closes.

§ 23. Common to all the recent logicians who have dealt seriously with the problem thus defined is the tendency to reduce

all the order-systems of mathematics to a form defined, so far as possible, in terms of a *few* simple and necessary "logical entities," and "fundamental hypotheses" about relational properties and about the objects whose relations are in question.

In all the older attempts to characterize the mathematical systems of an orderly type, great stress was laid upon the assumption of so-called "self-evident" *Axioms.* The example of Euclid in his Geometry, and the Aristotelian logical theory regarding the necessity of founding all proof upon "immediate" certainties,—these were the paramount influences in determining this tendency. But the more the logician considers the so-called "self-evident" principles of the older mathematical statements, the more reason does he see to condemn self-evidence as in itself a fitting logical guide. *When we call an assertion self-evident we generally do so because we have not yet sufficiently considered the complexity of the relations involved.* And many propositions have been supposed to be self-evident truths that upon closer acquaintance have turned out to be decidedly inexact in their meaning, or altogether incorrect.

In two cases, in the foregoing discussion, we have had occasion to indicate for logical purposes how inadequate the older assumptions regarding the axioms of mathematics and other sciences have been. The first case was presented to us by the presupposition of induction, to the effect that the realm of the objects of possible experience has in any of its definable collections of fact *a determinate constitution.* In mentioning this presupposition in § 10, we stated that it is not self-evident. In § 19 this presupposition appeared in the form of the postulate: *That there are Individuals.* The substantial identity of the two postulates appears upon due reflection. But, as we remarked (in § 19), the postulate: That there are individuals, is too complex to be self-evident, although, upon the other hand, a study of the conception of an individual led us to the assertion, not very fully discussed in this sketch, that this postulate is indeed *at once pragmatic and absolute.* As we said, in our former passage, the principle in question has metaphysical aspects that cannot here be discussed.

At all events, however, we gain, and we do *not* lose, by regarding the postulate of individuality not as "self-evident" but as the expression of an extremely complex, but at the

same time fundamental demand of the rational will,—a demand without which our activity becomes rationally meaningless.

The other case of a so-called "axiom" was mentioned in § 18, where we spoke of the principle: That things equal to the same thing are equal to each other. We gain instead of losing when this principle no longer seems self-evident, because we have come to observe that it involves a *synthesis* of the logically independent characters, transitivity and symmetry,—a synthesis which always needs to be justified, either by experience or by definition, or by demonstration, or finally, if that is possible, by the method which we have already applied in dealing with the concepts of class and of relation.

As a fact, therefore, most modern investigators of the Theory of Order have abandoned the view that the fundamental types of order can be defined in terms of merely "self-evident" axioms. These investigators have therefore come to be divided, largely, into two classes: (1) those who, in company with the Pragmatists, are disposed to admit a maximum of the empirical and the contingent into the theory of order; and (2) those who are disposed, like the present writer, to regard the fundamental principles of logic as sufficient to require the existence of a realm of ideal, *i.e.* of possible objects, which is infinitely rich, which contains systems such as the order-system of the numbers, and which conforms to laws that are in foundation the same as the laws to which one conforms when he distinguishes between *yes* and *no*, and when he defines the logical properties of classes and relations.

The writers of the first class would maintain, for instance, that whether or no such distinctions as that between *yes* and *no* have a necessary validity over and above that which belongs to physical objects, such systems as the *ordinal whole numbers* are simply hypothetical generalizations from experience, are empirically known to be valid so far as our process of counting extends, and are regarded in mathematics as absolute, so to speak, *by courtesy*. The field within which such logical empiricists very naturally find their most persuasive instances, is the field presented by geometrical theories. Geometry is a field in which purely logical considerations, and very highly contingent physical facts and relations, have been, in the past, brought into a most extraordinary union, which only recent research has begun to disentangle. Is Geometry at bottom a physical

science? Or is it rather a branch of pure logic, the discussion of an order-system or order-systems that possess a logically ideal necessity? The modern discussion of the Principles of Geometry has indeed greatly emphasized the enormous part that a purely logical Theory of Order plays in the development of geometry. But such a theory depends after all upon assumptions. Some of these assumptions, such as the famous Euclidian postulate regarding parallels, appear to some of the writers in question to have an obviously empirical foundation, as contingent as is the physical law of gravitation, and as much subject to verifications which are only approximate as that law is.

Over against these logical empiricists there are those who, however they analyze such special cases as that of geometry, agree with Mr. Bertrand Russell (in his *Principles of Mathematics*) in viewing the pure Theory of Order as dependent upon certain "logical constants." Such "logical constants" Mr. Russell assumes to be fundamental and inevitable facts of a realistically conceived world of purely logical entities, whose relation to our will or activity Mr. Russell would indeed declare to be factitious and irrelevant. Given the "logical constants," Mr. Russell regards the order-systems as creatures of definition; although, from his point of view, definition also appears to be a process by which one reports the existence, in the logician's realm, of certain *beings*, namely, classes, relation, series, orders, of the degrees of complexity described in our foregoing sketch. The Theory of Order for Mr. Russell is the systematic characterization of these creatures of definition. It asserts that the properties of these systems follow from their definitions. And pure mathematics consists of propositions of the type "p implies q," propositions p and q being defined, in terms of the "logical constants," and *so*, that, whatever entities there be (Mr. Russell's "variables") which are defined in terms of proposition p, are also such that proposition q holds of them. In the main Mr. Russell's procedure carries out with great finish ideas already developed by the school of Peano. Mr. Russell's doctrines serve, then, as examples of logical opinions which are not, in the ordinary sense, empiristic.

But the "burning questions" already mentioned as prominent in recent logical theory have shown how difficult it is to make articulate the theory of Mr. Russell, the somewhat similar

position of Frege in Germany, and the methods of the school of Peano, without making more "pressing" than ever the question as to *what* classes, series, order types, and systems are to be regarded as unquestionably existing in the world which the theory of order studies, when it abstracts from physical experience and confines itself to the entities and systems of entities which can be defined solely in terms of the "logical constants." There is no doubt of the great advance made in recent times by the writers of this school in actually working out the deductive consequences of certain postulates, when these are once used for the purpose of defining a system. And every such working out is indeed a discovery of permanent importance for the theory of order. To define, for instance, what are called ideal "space-forms," upon the basis of principles more or less similar to, or more or less general than, the postulates of Euclid, is to reach actual and positive results valid for all future Theory of Order. But as the present state of the Theory of Assemblages shows, serious doubts may rise in any one case as to whether such definitions and postulates do not involve latent contradictions, which render the theories in question inadequate to tell us what order-systems are indeed the necessary ones, and what the range of those entities is whose existence can be validated by considerations as fundamental as those which we have already used in speaking of classes and relations in general.

§ 24. One method of escape from the difficulties thus suggested is a way that, in principle, was pointed out a good many years since by an English logician, Mr. A. B. Kempe. In the year 1886, in the *Philosophical Transactions of the Royal Society*, Mr. A. B. Kempe published a memoir on the "Theory of Mathematical Form," in which, amongst other matters, he discussed the fundamental conceptions both of Symbolic logic and of Geometry. The ideas there indicated were further developed, by Mr. Kempe, in an extended paper "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," in the *Proceedings of the London Mathematical Society* for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. Kempe's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest.

In 1905 the present writer published, in the *Transactions of the American Mathematical Society*, a paper entitled "The Relation of the Principles of Logic to the Foundations of Geometry." This paper attempts (1) to show that the principles which Mr. Kempe developed can be stated in a different, and as the author believes, in a somewhat more precise way; and (2) that the principles in question, namely the principles which are involved in any account of the nature of logical classes and their relations, are capable of a restatement in terms of which we can define an extremely general order-system. This order-system is the one which Mr. Kempe had partially defined, but which the present author's paper attempted to characterize and develop in a somewhat novel way. The thesis of that paper, taken in conjunction with Mr. Kempe's results may be restated thus:—

Both classes and propositions are objects without which the logician cannot stir a step. Their relations and laws have therefore, in the foregoing sense, an absolute validity. But, if we state these relations as laws in a definite way, and if we thereupon define a further principle regarding the existence of certain logical entities which in many respects are similar to classes and propositions,—a principle not heretofore expressly considered by logicians,—we hereupon find ourselves forced to conceive the existence of a system called, in the paper of 1905, "The System Σ." *This system has an order which is determined entirely by the fundamental laws of logic, and by the one additional principle thus mentioned.* The new principle in question is precisely analogous to a principle which is fundamental in geometrical theory. This is the principle that, between any two points on a line, there is an intermediate point, so that the points on a line constitute, for geometrical theory, *at least a dense series.* In its application to the entities of pure logic this principle appears indeed at first sight to be extraneous and arbitrary. For the principle corresponding to the geometrical principle which defines dense series of points, does not apply at all to the logical world of propositions. And, again, it does not apply with *absolute* generality to the objects known as classes. But it *does* apply to a set of objects, to which in the foregoing repeated reference has been made. This set of objects may be defined as, "certain possible modes of action that are open to any rational being who can act at all, and who can also reflect

upon his own modes of possible action." Such objects as "the modes of action" have never been regarded heretofore as logical entities in the sense in which classes and propositions have been so regarded. But in fact our modes of action are subject to the same general laws to which propositions and classes are subject. That is :—

(1) To any "mode of action," such as "to sing" or "singing" (expressed in English either by the infinitive or by the present participle of the verb) there corresponds a mode of action, which is the *contradictory* of the first, for example "not to sing" or "not singing." Thus, in this realm, to every x there corresponds *one*, and essentially *only* one, $\bar{x}$.

(2) Any pair of modes of action, such for instance as "singing" and "dancing," have their "logical product," precisely as classes have a product, and their "logical sum," again, precisely as the classes possess a sum. Thus the "mode of action" expressed by the phrase: "To sing and to dance" is the logical product of the "modes of action" "to sing" and "to dance." The mode of action expressed by the phrase, "Either to sing or to dance," is the logical sum of "to sing" and "to dance." These logical operations of addition and multiplication depend upon triadic relations of modes of action, precisely analogous to the triadic relation of classes. So then, to any x and y, in this realm, there correspond $x\,y$ and $x+y$.

(3) Between any two modes of action a certain dyadic, transitive and not totally non-symmetrical relation may either obtain or not obtain. This relation may be expressed by the verb "implies." It has precisely the same relational properties as the relation $\prec$ of one class or proposition to another. Thus the mode of action expressed by the phrase, "To sing *and* to dance," *implies* the mode of action expressed by the phrase "to sing." In other words "Singing *and* dancing," implies "singing."

(4) There is a mode of action which may be symbolized by a o. This mode of action may be expressed in language by the phrase, "to do nothing," or "doing nothing." There is another mode of action which may be symbolized by 1. This is the mode of action expressed in language by the phrase "to do something," that is, to act positively in any way whatever which involves "*not doing nothing*." The modes of action 0 and 1 are contradictories each of the other.

In consequence of these considerations, *the modes of action are a set of entities that in any case conform to the same logical laws to which classes and propositions conform.* The so-called "Algebra of Logic" may be applied to them. A set of modes of action may therefore be viewed as a system within which the principles of logical order must be regarded as applicable.

Now it would indeed be impossible to attempt to define with any exactness "the *totality* of all possible modes of action." Such an attempt would meet with all the difficulties which the Theory of Assemblages has recently met with in its efforts to define certain extremely inclusive classes. Thus, just as "the class of all classes" has been shown by Mr. Russell to involve fairly obvious and elementary contradictions, and just as "the greatest possible cardinal number" in the Cantorian theory of cardinal numbers, and equally "the greatest possible ordinal number" have been shown to involve logical contradictions, so (and unquestionably) the concept of the "totality of all possible modes of action" involves a contradiction. There is in fact no such totality.

On the other hand, it is perfectly possible to define a certain set, or "logical universe" of modes of action such that all the members of this set are "possible modes of action," *in case* there is some rational being who is capable of performing some one single possible act, and is also capable of noting, observing, recording, in some determinate way every mode of action of which he is actually capable, and which is a mode of action whose possibility is *required* (that is, is made logically a necessary entity) by the *single* mode of action in terms of which this system of modes of action is defined. Such a special system of possible modes of action may be determined, in a precise way, by naming *some one* mode of action, which the rational being in question is supposed to be capable of conceiving, and of noting or recording in some reflective way any mode of action once viewed as possible. The result will be that any such system will possess its own logical order-type. And some such system must be recognized as belonging to the realm of genuinely valid possibilities by any one who is himself a rational being. The order-type of this system will therefore possess a genuine validity, a "logical reality," which cannot be questioned without abandoning the very conception of rational activity itself. For the question is not whether there exists any being

who actually exemplifies these modes of activity in the same way in which "singing and dancing" are exemplified in our human world. The logical question is whether the special sets of modes of action whose logical existence as a set of possible modes of action is required (in case there is any one rational being who can conceive of any one mode of action), is a genuinely valid system, which as such has logical existence.

Now the logical system of such modes of action illustrates a principle, which, as just admitted, *does not apply to the Calculus of Propositions.* Nor does this principle apply, with complete generality, to the Calculus of Classes. But what we may here call the Calculus of Modes of Action, while it makes use of all the laws of the Algebra of Logic, also permits us to make use of the principle here in question, and in fact, in case a system of modes of action, such as has just been indicated, is to be defined at all, *requires* us to make use of this principle. The principle in question may be dogmatically stated thus: "If there exist two distinct modes of action p and r, such that $p \mathrel{-\!\!<} r$, then there always exists a mode of action q such that $p \mathrel{-\!\!<} q \mathrel{-\!\!<} r$, while p and q are distinct modes of action and q and r are equally distinct." This principle could be otherwise stated thus "for any rational being who is able to reflect upon and to record his own modes of action, if there be given any two modes of action such that one of them implies the other, there always exists at least one determinate mode of action which is implied by the first of these modes of action and which implies the second, and which is yet distinct from both of them." That this principle holds true of the modes of action which are open to any rational being to whom any one mode of action is open, can be shown by considerations for which there is here no space, but which are of the nature heretofore repeatedly defined in this paper. For the question is not whether there actually lives any body who actually does all these things. *That*, from the nature of the case, is impossible. The question is as to the definition of a precisely definable set of modes of action. And this principle holds for the Calculus of possible modes of action, because, as can be shown, the denial of such a principle for a rational being of the type in question, would involve self-contradiction.

Now the consideration developed by Kempe, and further elaborated in the paper of 1905, before cited, may be applied,

and in fact must be applied to the order-system of such a determinate realm of modes of action. Such a realm is in fact of the form of the foregoing system Σ. A comparison of Kempe's results with the newer results developed in the author's later paper would hereupon show :—

(1) That the members, elements, or "modes of action" which constitute this logically necessary system Σ exist in sets both finite and infinite in number, and both in "dense" series, in "continuous" series, and in fact in all possible serial types.

(2) That such systems as the whole number series, the series of the rational numbers, the real numbers, etc., consequently enter into the constitution of this system. The arithmetical continuum, for instance, is a part of the system Σ.

(3) That this system also includes in its complexities all the types of order which appear to be required by the at present recognized geometrical theories, projective and metrical.

(4) That the relations amongst the logical entities in question, namely the *modes of action*, of which this system Σ is composed, are not only dyadic, but in many cases polyadic in the most various way. Kempe, in fact, shows with great definiteness that the triadic relations of ordinary logic, which are used in defining "sums" and "products," are really dependent upon tetradic relations into which 0 or 1, one or both may enter. In addition to these tetradic relations the logical order-system also depends for some of its most remarkable properties upon a totally symmetrical tetradic relation that, in the sense described in § 18, is transitive by pairs. These special features of the system of logical entities are here mentioned for the sake of merely hinting how enormously complex this order-system is. The matter here cannot be further discussed in its technical details. The result of these considerations is *that it at present appears to be possible to define, upon the basis of purely logical relations, and upon the basis of the aforegoing principles concerning rational activity, an order-system of entities inclusive not only of objects having the relation of the number system, but also of objects illustrating the geometrical types of order, and thus apparently including all the order-systems upon which, at least at present, the theoretical natural sciences depend for the success of their deductions.*

And so much must here serve as a bare indication of the problems of the Theory of Order, problems which, at the present

day, are rapidly undergoing reconsideration and which form an inexhaustible topic for future research. Of the fundamental philosophical importance of such problems no student of the Categories, no one who understands the significance of Kant's great undertaking, no one who takes Truth seriously, ought to be in doubt. The Theory of Order will be a fundamental science in the philosophy of the future.

THE PRINCIPLES OF LOGIC

BY

LOUIS COUTURAT.

"Symbolic logic considered as a calculus has undoubtedly much interest on its own account; but this aspect has hitherto been too much emphasized at the expense of the aspect in which symbolic logic is merely the most elementary part of mathematics, and the logical prerequisite of all the rest."

B. RUSSELL.[1]

IT is not without some embarrassment that I appear in this volume as the representative of French philosophy; for, on the one hand, I entertain neither the aim nor the pretension of expressing the views of French philosophers on Logic, and, on the other, the theories I am going to present are due to authors of diverse nations, amongst which I regret not to be able to include the French.

Liberal as is the space which has been assigned to me, I cannot attempt to offer an historical sketch, however brief, of the modern theories of Logic, for in so doing I should run the risk of giving a superficial and even false idea of them, while, if I confined myself to the principal doctrines, I might be unjust to the authors of other systems. Moreover, this method of exposition has the defect of presenting theories under an individual form, and of thus assimilating them to works of art. But if Philosophy, and more especially Logic, is scientific in character and has objective value, the fundamental theories can and should be presented under an impersonal form. This is the best way, too, to bring out their unity, and to show that this unity is the collective and progressive work of many thinkers.

I shall not delay over the definition of Logic and the determination of its sphere. Logic is taken here in the traditional and classic sense, *i.e.* as the normative science of the

[1] "The Theory of Implication," in *Amer. Journal of Math.* vol. xxviii. p. 184.

formal laws of correct thinking. If this science (like many others) has made enormous progress in the nineteenth century, and has taken on a special form analogous to that of Algebra in Mathematics, this progress and this form have in no way changed its nature; on the contrary, they are conformable with its essence. It is natural and necessary that a formal science which abstracts from the contents of concepts in order to be able better to study their relations should employ formulae which constitute a special algorithm. This step is taken as soon as we give up reasoning from concrete examples, the matter of which might give rise to illusions as to the validity of the form, and represent the terms by letters: in this sense Aristotle was the first "logistician," the schoolmen imitated him, and the moderns have only developed and perfected this primitive symbolism. The terms quoted below are to be rejected on various grounds: *symbolic Logic*, because it lays too much weight on a secondary detail of notation;—*mathematical Logic*, because it prejudges the relations between Logic and Mathematics, and suggests a doubly false meaning: firstly, the application of Mathematics to Logic, when the latter ought to have its own independent laws; secondly, the application of Logic to Mathematics, which restricts the universal scope of Logic;—*algebraic Logic* or *Algebra of Logic*, because these also are ambiguous terms; moreover, they exaggerate the importance of the algebraic form of modern Logic, and by regarding Logic as a form of algebra reduce it to a special branch of the mathematical sciences, while the latter, on the contrary, are collectively dependent on Logic conceived as the art of reasoning. We prefer the term *algorithmic Logic*, which simply indicates the fact that logical laws give rise to a sort of calculus; or, better still, that of *Logistic*, which (having lost its ancient signification) suggests the same idea. But it must be clearly understood that these terms do not imply any deviation from or restriction of the traditional conception of Logic; they merely characterize the modern method of treating the old and enduring problems of classical Logic.

If we employ symbols, therefore, it is not because we attach any essential importance or mysterious virtue to them, but simply because they furnish a clearer, more precise and more plastic expression of fundamental notions and relations; moreover, the reader will thus become familiarized with the formulae

of Logistic, and will be prepared to study original works. We cannot, of course, quote all the different symbolizations, almost as numerous as the writers on the subject, which have been suggested; we have adopted one which is, properly speaking, not that of any one author, but which, owing to its analogy with mathematical symbolism, seems to us the most convenient. It approximates most closely to Schröder's. M. Peano's system is governed by the necessity of combining in the same formulae logical and mathematical symbols, hence he was obliged to give the former a different form. But this motive had a *particular* application in view; while for our theoretical and didactic exposition it behoves us, on the contrary, to lay stress on the formal analogy (though we must not lose sight of the differences) between the logical calculus and the algebraical calculus.

I. THE LOGIC OF PROPOSITIONS.

The old Logic began with the theory of concepts, or terms, because it was restricted to the study of the relations between concepts (*i.e.* judgments of attribution). Modern Logic prefers to take the proposition as its ultimate element.

We shall not attempt here to give a rigorously *logical* exposition of the principles of Logic. Such an exposition is very difficult in any science, but it would probably be impossible in Logic, for when we are dealing with the primary concepts of thought in general it is impossible to find any others by which these can be defined. What would be the good, for instance, of accepting the notion of *implication* as indefinable, and then going on to define the proposition as "every thing which implies itself"? Is not this a reversal of the natural order of ideas, and is it not clear that implication, a relation between two propositions, presupposes the notion of "proposition"? Paradoxical as it may appear, it is impossible to have a *logical* exposition of the principles of Logic: we are condemned in advance to a *petitio principii* or to a vicious circle. Instead of attempting to disguise these by an apparatus of forms which should distort or reverse the natural order of ideas, we prefer to admit them frankly from the beginning, without any idle logical vanity.

We must begin by admitting two fundamental concepts, *i.e.*

those of true and false:[1] these are concepts of quality, they qualify certain thoughts of ours. We may go on to define the *judgment* as a thought which can be qualified as true or false. The expression of a judgment is a proposition; it can be determined by the same qualifications.

It will be understood that we are here speaking of a concrete and fully determined thought, such as the thought of a particular fact: "It rains in this place at this moment," and not simply "It rains," which is an indeterminate and incomplete judgment. This example shows us at the same time that there are judgments without terms, without subject or attribute. A judgment is essentially *the assertion of a fact.* It is true if the fact is real, it is false if the fact is not real,[2] and since the fact (completely determined) is necessarily either real or not real the judgment must be either true or false.

The most fundamental relation which can exist between two propositions is that of *implication.* If A and B are the propositions, then "A implies B" means (1) "Either A is false or B is true." Of the four cases possible for any two propositions

(1) A true, B true; (2) A true, B false;
(3) A false, B true; (4) A false, B false;

the above implication excludes the second and the second only. That is to say, it is equivalent to saying (2) "It is false that A is true and B false." It is represented by the symbol: $A < B$ and, in speech, by the following: (3) "If A is true B is true," or "If A (then) B." We cannot regard this translation as a definition, for it takes the meaning of the conjunction "if" as known. But this meaning is a subtle one, and we must look at definition (1) given above for its precise definition.

But this definition (1) contains another conjunction (either—or); and if the formula (2) is preferred it contains the conjunction "and." In any case we have not been able to define the meaning of "implication" except as a function of one of these two notions: or, and. They are the two essential *combinations* into which propositions enter. They are called (from reasons of

[1] The definition of these two terms, or the inquiry into their signification, is a metaphysical or, if it be preferred, metalogical problem. The symbol for true is 1 and for false 0.

[2] To prevent any dispute over the expression "a not real fact," let us say, to please everybody, that the fact represented "*does not exist.*"

analogy, the *logical sum* and the *logical product*, and the logical "operations" which consist in forming these combinations are called *logical addition* and *multiplication*.

The *logical sum* of two propositions A, B, which is written A + B, is a proposition which affirms that one at least of these two propositions is true. It is expressed as follows: "A or B is true," or "A or B." It excludes nothing except the case in which A and B are both false. It is equivalent, then, to saying: "It is false that A and B are both false." We can define it by means of the conjunction *and*, that is to say, by multiplication. The *logical product* of two propositions A, B, which is written A × B, A. B, or AB, is a proposition which affirms that A and B are both true. It is written: "A is true and B is true," or "A and B."[1] It only admits the first case of the table given above, and excludes the three others. The product of two propositions is sometimes called their *simultaneous affirmation*, and the sum their *alternative affirmation* (or their *alternation*). The word "simultaneous" is not used in its temporal sense here, but in the sense it has in mathematics when we speak of *simultaneous* equations (*i.e.* remaining true for the same unknowns, and being verifiable "at the same time" by them).

The *equality* or *equivalence* of two propositions is defined as a function of their implication. The propositions A and B are said to be *equal* (or *equivalent*) if A implies B and B implies A; this is written[2]

$$(A = B) = (A < B)\ (B < A).$$

The equivalence of two propositions in no way implies their identity in meaning. We get a similar equivalence every time that the converse of a theorem is true: for a theorem signifies that its hypothesis implies its thesis, and its converse signifies that its thesis implies its hypothesis.[3]

[1] If this explanation set up as a definition it would be vicious, for we have defined "and" by means of "and."

[2] This formula, considered as a definition of equality seems to constitute a vicious circle, for it has as its copula the relation = which was to be defined. To which it must be answered that this copula is *equality by definition*, which is distinct from the real relation of *equality* (cf. chap. v. Methodology).

[3] Example: Every triangle which has two equal sides has two equal angles. Converse: Every triangle which has two equal angles has two equal sides. The two propositions: This triangle has two equal sides; This triangle has two equal angles, are equivalent but have not at all the same sense. The same is true of the corresponding concepts: isosceles triangle and equilateral triangle.

By definition an implication is true if its first member is false or its second member true. Thus the false may imply anything whatever and the true be implied by anything whatever. This gives us:

$$o < x, \qquad x < 1$$

whatever x may be.[1] We have then, as a particular case,

$$o < 1.$$

"The false implies the true," a paradox which easily finds its explanation in the above given definition. It follows from this that if a proposition implies the false it is equal to it, and if it implies the true it is equal to it:

$$(x < o) = (x = o), \qquad (1 < x) = (1 = x).$$

These equivalences will be confirmed by common sense.

To sum up: o and 1 constitute two particular *values* of propositions, and the *only* ones that they can have. If we consider an implication, we shall see that it can only present these four combinations of values:

$$o < o, \qquad o < 1, \qquad 1 < o, \qquad 1 < 1;$$

of these it is only false for the third, for the three others it is true.

Negation is an operation or relation fundamental for logic. It may be defined by means of multiplication and addition, and the notions of true and false as follows:

Given a proposition A, its negation is a proposition A′, so that we have at the same time

$$A \times A' = o, \qquad A + A' = 1.$$

In systematizing the axioms of Logic the *existence* of this negation has to be postulated. We can then demonstrate that it is unique, *i.e.* that all the possible negations of a given proposition are equivalent to one another. Let the negation of A be symbolized by A′.[2] We can then in addition demonstrate

[1] This formal correctness may seem to define o and 1, but in that case their existence must be postulated. "There exists an entity o which implies anything whatever, and an entity 1 which is implied by anything whatever" (cf. chap. v. Methodology).

[2] If we want to indicate the negative of a complex expression we put the latter in brackets and place the negative sign ′ after the bracket.

the law of double negation, *i.e.* that the negation of non-A is A. This results immediately from the symmetry of the relations which determine negation. A is in relation to A′ what A′ is in relation to A. This truth is expressed in the aphorism of ordinary speech: "Two negations are equal to an affirmation." Inversely, if we take negation for our primary concept we can use it to define addition by means of multiplication, or the other way round. In fact, from the definition of negation we get the two formulae of De Morgan:

$$(A+B)' = A'B', \qquad (AB)' = A' + B';$$

i.e. "The negation of the sum of two propositions is the product of their negations"; "the negation of their product is the sum of their negations." These two formulae can be verified by simple common sense: to deny that two propositions are true is to affirm that one or the other is false: to deny that one or the other is true is to affirm that both are false.[1]

If, then, we admit negation and multiplication, we can define addition by means of the formula:

$$A + B = (A'B')'.$$

"The sum of two propositions is the negation of the product of their negations."[2] Or inversely, we can define multiplication by means of addition:

$$A\ B = (A' + B')'.$$

We will leave to the reader the task of interpreting this last formula in words. We see that according to whether we adopt one system of axioms or another, the same logical formulas appear as principles, as definitions or as theorems. But in every case one truth emerges from these inquiries into the axioms of Logic, *i.e.* that the *principle of identity*, the *principle of contradiction*, and *the principle of excluded third* are three verities independent of one another. In fact, these three

[1] We have just been tacitly applying two important formulae of the calculus of propositions:

$$A = (A = 1), \qquad A' = (A = 0).$$

"A proposition A is equal to the assertion: A is true; its negation is equal to the assertion: A is false."

[2] To say that A or B is true is to deny that "A and B" is false; it is to say that A and B cannot be false together.

principles, applied to propositions, may be translated by the following formulae:

$$A < A.$$

"A implies A" or "if A is true A is true."

$$AA' = 0.$$

"A and not-A cannot be true at the same time."

$$A + A' = 1.$$

"Either A is true or not-A is true." Now we recognize in the two last the formulae which we used to define negation. Moreover, it would be impossible to deduce these from the principle of identity, because they assume the concept of negation, while the principle of identity is independent of it. Whether we define negation or take it as our fundamental notion, in either case we introduce a new element or a postulate. Hence all attempts to reduce the three "laws of thought" to a single one are formally impossible.

But this is not all: these three principles alone are not sufficient to justify the smallest deduction, and one must add to them other principles which are independent of them. The most important of these are—

The principle of syllogism [1]:

$$(A < B)\ (B < C) < (A < C),$$

"If A implies B, and if B implies C, A implies C";

and the *principle of deduction*:

"If A implies B, and if A is true, then B is true (and we can affirm this independently)."

We should be tempted to express this principle by the formula:

$$(A < B)\ (A = 1) < (B = 1),$$

but this formula is again an implication from which we cannot extract the thesis $(B = 1)$ in order to affirm it separately except in virtue of this principle itself. And here we have a striking proof of the necessary limitation of symbolism when it is a question of formulating principles. It will be remarked that in every practical application of the principle of syllogism (or of

[1] This is the principle of the *hypothetical* (not of the categorical) syllogism, the terms of which are propositions, so that the premisses and the conclusion are *hypothetical* judgments.

any other principle which contains an implication) we tacitly invoke the principle of deduction, for we should not be content to affirm the implication which binds the conclusion to the premisses; what we desire is to be able to affirm the conclusion independently of the premisses when the latter are true. Now this is only permitted in virtue of the principle of deduction. Speaking generally, no deduction is possible without this principle, and it is this which justifies its name.

Once the preceding principles and definitions are admitted, we can demonstrate the laws of multiplication and addition, which we have till now been content to enunciate.

(1) *The law of commutation*:

$$AB = BA. \qquad A + B = B + A.$$

(2) *The law of association*:

$$(AB)C = A(BC). \qquad (A + B) + C = A + (B + C).$$

(3) *The law of distribution*:

$$(A + B)C = AC + BC. \qquad AB + C = (A + C)(B + C).$$

All these formulae, with the exception of the last, are common to the logical and the algebraical calculuses. The latter only is peculiar to the logical calculus; thanks to it and more generally to the perfect reciprocity (*duality*) of the formulae of addition and multiplication, the logical calculus exhibits a symmetry which the algebraical calculus does not possess.

(4) *The law of tautology*:

$$AA = A. \qquad A + A = A.$$

(5) *The law of absorption*:

$$A + AB = A. \qquad A(A + B) = A.$$

(6) *The law of simplification*:

$$AB < A. \qquad A < A + B.$$

(7) *The law of composition*:

$$(A < B)(A < C) < (A < BC). \qquad (B < A)(C < A) < (B + C < A).$$

The laws of tautology and of absorption permit us to simplify the terms of a sum or of a product. The laws of simplification and of composition permit us either to extract from a sum or a product a consequence which is contained within them, or to combine two implications into a single one.

Here are some analogous formulae which are derived from them:

$$(A<B)<(AC<BC). \qquad (A<B)<(A+C<B+C).$$

"The same factor or the same summand may be added to the two members of an implication:"

$$(A<B)(C<D)<(AC<BD). \quad (A<B)(C<D)<(A+C<B+D).$$

"Two implications may be combined, member by member, either by multiplication or by addition."

We can now demonstrate some important formulae which permit the transformation of an implication into an equation:

$$(1)\ (A<B)=(A=AB). \qquad (A<B)=(A+B=B).$$
$$(2)\ (A<B)=(AB'=0). \qquad (A<B)=(A'+B=1).$$

These last formulae are the most interesting: "to say that A implies B is to deny that A and not-B are true at the same time; or it is to affirm that not-A or B is true."[1] We rediscover here by means of the calculus the *verbal* definition which we gave of implication; but there is no *petitio principii*, for we did not make use of the verbal definition, and we took implication as our fundamental principle.[2]

To calculate the two values 0 and 1, we have the formulae:

$$0 . 0=0, \quad 0 . 1=0, \qquad 0+0=0,$$
$$1 . 1=1, \qquad 0+1=1, \quad 1+1=1,$$

from which it results that 0 and 1 are the negations one of another, since they fulfil the formal definition of negation:

$$0 . 1=0. \qquad 0+1=1.$$

"The false is the negation of the true and inversely." And this is confirmed by common sense. Thus we sometimes arrive at truisms by way of paradoxes.

[1] The two formulae are equivalent, moreover; in the case of the de Morgan formulae AB' is the negation of $A'+B$, and 0 the negation of 1.

[2] According to Russell ("The Theory of Implication" in *Amer. Journal of Math.* vol. xxviii. p. 161) the essence of implication consists in this. "What is implied by a true proposition is true": and this is why implication can be used in deduction. Now this signifies that, in an implication, it is impossible that the first member should be true and the second false ($AB'=0$), or, again, that either the first member is not true or the second is true: ($A'+B=1$). In virtue of the foregoing formulae, the principle of identity ($A<A$) is equivalent to the principle of contradiction ($AA'=0$), and to the principle of excluded middle ($A+A'=1$). But this is not a reduction of the three principles to a single one, for the equivalences are based on the definition of negation, *i.e.* upon one of the last two principles themselves.

The formulae that we have just enumerated permit us first to reduce implications to equations, then to reduce equations of any form whatever to equations of which the second member is 0 or 1 ; finally to combine together these equations by addition or multiplication, for from the law of composition there results as a particular case the following formulae:

$$(A=0)\,(B=0)=(A+B=0). \qquad (A=1)\,(B=1)=(AB=1).$$

Thus we can treat a logical problem by reducing all its data or premisses to a single *equation*, which may be resolved in relation to one or several unknowns, or from which we can draw all the consequences desired by the formal rules. The algebra of Logic, then, like the ordinary algebra, ends in a theory of equations. We may also study in it the functions of Logic, their development and their transformation. We shall not insist on this mathematical and purely formal side, which is of more interest for mathematicians than for philosophers, but will confine ourselves to indicating that the principles already quoted can serve as a foundation for a real algebra which has its own laws, sometimes analogous to those of arithmetical algebra, and sometimes very different.[1]

We prefer to give some formulae which correspond to important and usual types of reasoning. For example, from the formula of transformation:

$$(A<B)=(AB'=0)$$

the *law of contraposition* can easily be deduced:

$$(A<B)=(B'<A').$$

"If A implies B, not-B implies not-A ; and reciprocally."[2] This is the formula of *reductio ad absurdum* : to prove that in a theorm the hypothesis A implies the thesis B we suppose that B is false, and deduce from that the falsity (*i.e.* the negation) of A.

We may also deduce the *law of transposition*, of which contraposition is a special case:

$$(AB<C)=(AC'<B'). \qquad (C<A+B)=(B'<A+C').$$

[1] See our opuscule *L'Algèbre de la Logique* (Paris: Gauthier-Villars, 1905). See also *Lezioni di Algebra della Logica*, by A. Delke (Napoli: R. Accademia, 1907).

[2] This "reciprocally" is indicated in the sign =, which is equivalent to two inverse implications.

The two implications, in fact, the equality of which is affirmed, are each equivalent to:

$$AB C' = 0, \qquad A'B'C = 0.$$

The first of these equations signifies briefly that the three propositions A, B, and not-C are *incompatible*; we can therefore deduce from it either that A and B imply C, or that A and not-C imply not-B, or that B and not-C imply not-A. Similarly for the second equation. We see the advantage of reducing an implication to the more symmetrical form of what the English call an *inconsistency*, *i.e.* an equation of which the second member is nought.

We have again, still in the same way,

$$(A < B + C) = (AC' < B).$$

"If A implies B or C, A and not-C imply B, for the two implications are equal to:

$$AB'C' = 0."$$

This is the very commonly occurring type of an *alternative syllogism*: "If A is true, B or C is true: but C is false (and A true), therefore B is true."

If we apply this transformation to the formula:

$$(A < B) < (A = 0) + (B = 1),$$

we can deduce from it:

$$(A < B)\,(A = 1) < (B = 1),$$

the formula of the *direct hypothetical syllogism* (*modus ponens*): "If A implies B, and if A is true, B is true."

We can also deduce from it:

$$(A < B)\,(B = 0) < (A = 0),$$

the formula of the *inverse hypothetical syllogism* (*modus tollens*): "If A implies B, and if B is false, A is false."

Another type of reasoning by *reductio ad absurdum* has been brought forward by Vailati; his formula is:

$$(A < A') < (A = 0).$$

"If A implies (has as its consequence) its own negation, A is false." And indeed:

$$(A < A') = (AA = 0) = (A = 0).$$

Inversely: "If not-A implies A, A is true."

$$(A' < A) < (A = 1);$$

i.e. $$(A' < A) = (A + A = 1) = (A = 1).$$

These forms of reasoning, paradoxical but perfectly correct, have sometimes been employed by mathematicians.

II. PROPOSITIONAL FUNCTIONS.

So far we have been considering propositions as the *elements* of reasoning, and have not troubled to analyse their contents. We have been reversing the traditional order which begins by the theory of terms or concepts. We must now penetrate within propositions in order to distinguish their constitutive elements. Hitherto we have considered each proposition as an individual, as a particular and perfectly determined being. Nevertheless the mere fact that we were reasoning about *any* propositions *whatsoever* and employing algebraic symbols obliged us to deal with *general* or *indeterminate* propositions. For example, if we formulate *the law of simplification*:

$$AB < A,$$

we affirm this implication of all the propositions which can be substituted for A and B. We suppose, indeed, that A and B are singular and determined propositions; but owing to the fact that they can be any propositions whatever, the implication $AB < A$ becomes general and indeterminate, because the terms A and B which figure in it are in reality *variables*.

What is a variable? This question has been discussed with much depth and subtlety by *Frege* and *Russell*. A variable is an indeterminate term for which we can substitute any determinate term (belonging to a certain class): these determinate terms are called the *values* of the variable.[1] Speaking still more exactly, a variable is simply the sign of an *empty place*, a missing term which can be supplied; but under this condition, that for the same letter the same value shall everywhere be substituted (for example, in $AB < A$ we can substitute any values whatsoever in place of A and B, but we must always substitute the same value for the second A as for the first). A determinate proposition becomes indeterminate when one of its terms is suppressed or replaced by a variable, *i.e.* when it is assumed that any value whatever can be put in its place. An indeterminate proposition may contain one or several variables.

Speaking generally, we call every expression which contains one or more variables a *function*, *e.g.* $A + A'B + A'B'C$ is

[1] Values which are called, in contradistinction, constant or fixed.

a function of A, B, C, if we consider these three letters as variables. But when a logical function has the form of a proposition, it is called a *propositional function.* Thus what we call an *indeterminate proposition* is not really a proposition, but a *propositional function.* It is not a proposition, for under its indeterminate form it is neither true nor false; it has no meaning. It *becomes* a proposition when fixed values are substituted for the variables; it then acquires meaning, and consequently has a logical determinate value (true or false).[1]

A propositional function is as it were a matrix for propositions,—it engenders as many propositions as there are values assigned to its variables. A proposition can only have one value, *i.e.* false or true, but a propositional function can have as many distinct values as its variables can have values or systems of values. Thus the *general* implication $A<B$ is true or false, according to circumstances; we know that it is only false in the case when A is true and B false. But there are propositional functions which are *always true*; such are all the logical formulae which we have already quoted, *e.g.* the law of simplification: $AB<A$. From the very fact that it is a law of formal logic, it is true for all the values which can be assigned to A and B. Similarly, there are propositional functions which are *always false*: notably those which deny or violate a logical law. These are respectively *formal truths* and *falsities*, for their truth (or falsity) depends on their form and not on their matter; if it depended on their matter it would be variable.[2] It is particularly important to distinguish sharply between *formal implications* which contain variables and the *material implications* which we have not yet considered, and which hold good between singular propositions. All mathematical truths are *formal* implications, and, indeed, these are the implications which are generally employed, even in discussions of the most diverse kinds; for we pass general far more frequently than particular judgments.

To indicate symbolically that a propositional function is

[1] Because these concepts of *variable* and *function* are more familiar to mathematicians than to logicians, it must not be concluded that we have here an illegitimate intermixture of mathematical with logical concepts. It is easy to see, from their generality, that these concepts must be common both to Logic and to Mathematics.

[2] If we compare, for example, the *always true* formula $AB<A$ with the formula $AB<C$, the only difference between them is that in the first $C=A$ in virtue of its form.

always true, *i.e.* true for any values whatsoever of the variables, we will write these variables as indices after the copula.

For example:[1]

$$0 <_x x, \quad x <_x 1,$$
$$(x+y)' =_{x,y} x'y'.$$

The presence of these indices is sufficient to indicate that the formulae represent functions ("general" propositions) and not singular propositions. In order to express that a function F (of two variables, x, y) is always true or always false, we will write[2]

$$F(x, y) =_{xy} 1, \qquad F(x, y) =_{x,y} 0.$$

Similarly:

$$F(x, y) <_{x,y} \Phi(x, y)$$

signifies: "Whatever may be the values of x, y, the function F implies the function Φ."

The theory of propositional functions is based on some special axioms which we will content ourselves with merely stating, for their justification would demand long and subtle arguments upon which we cannot here enter.[3] "That which is true of all is true of any"; this evidently follows from the meaning of the word *all*. If a function $F(x)$ is true for all values of the variable x, it is true of any one among them singly. But the converse of this axiom is equally necessary: "What is true of any is true of all." That is to say: if a function $F(x)$ is true for *any* value *whatsoever* of x, it is true of *all* values of x. We shall recognize the importance and necessity of this axiom if we note that it is the hidden nerve of all mathematical reasoning. To demonstrate a theorem we argue concerning *any* number, or *any* triangle or circle, but they are always particular; and we conclude from them that the theorem is true for *all* numbers, *all* triangles, or *all* circles. This is not a case of induction, as certain empirical logicians have believed; it is a deduction based on the preceding axiom, *i.e.* on the fact

[1] It is important to note that in these formulae the symbols 1 and 0 no longer stand simply for the true and the false but for the "always true" and the "always false." Strictly speaking, we ought to employ a fresh symbol, but the indices are sufficient to prevent any confusion.

[2] For lack of this symbolism we were obliged to say above: "0 implies x, and x implies 1, *whatever x may be*," in order to specify clearly the generality or indetermination of x.

[3] See Russell, "Mathematical Logic as based on the Theory of Types," in *Amer. Journal of Math.* vol. xxx.

that if the demonstration holds good of *any individual whatever* it holds good for all. (The principle of sufficient reason has sometimes been invoked to justify this method of demonstration.) "What is true of all is true of one in particular." This axiom is the well-known principle of all particular applications of a general theorem. If $F(x)$ is true for all values of x it is true in particular for the value a, or the value b, etc. This principle permits the deduction of a true proposition from a true propositional *function*, *i.e.* to pass from the general truth $F(x) = {}_x1$ to the particular truth $F(a) = 1$. It will be noted that 1 has not the same signification in these two formulae; in the first it is accompanied by the index x, but not in the second.

There are other axioms, the character of which is more strictly algorithmic, *i.e.* which have more properly the effect of rendering possible certain combinations or transformations of the calculus.

We can establish a *hierarchy* between propositional functions, according to whether they belong to more or less complicated *types*. (It is no longer a question of the *number* of the variables according to which they can be classed, but of a property analogous to the *degree* of algebraic equations.) An elementary proposition holds good of *individuals*, *i.e.* of concrete and determinate objects which can be named or pointed out one by one. If we replace one or more of these individuals by variables we obtain a propositional function of the first order. But these propositional functions of the first order become in their turn objects of thought on which judgments, true or false, can be passed. If we replace the propositional functions of the first order in the latter by variables we obtain propositional functions of the second order, *i.e.* of which the variables are (or rather represent) functions of the first order. Similarly, we can successively conceive functions of the third and fourth order and so on, of which the variables are respectively functions of the preceding order.

This *theory of types* permits the resolution of certain contradictions or of certain paradoxes which arise when the expression "all possible propositions" is carelessly used; for then the proposition enunciated seems to become its own object, and we have a sort of vicious circle. The type of these paradoxes is the famous sophism of Epimenides. "Epimenides says that all Cretans are liars: now Epimenides is a Cretan,

therefore Epimenides is a liar, therefore the Cretans are not liars," etc. The sophism lies in the fact that the assertion of Epimenides ("the Cretans are liars" or more simply, "I am a liar") is supposed to hold good of itself (I lie even in saying that I lie). Epimenides ought correctly to have said: "All the propositions *of the first order* that I assert are false." Now this is a proposition of the second order which might be incontrovertibly true. We cannot speak of *all* the propositions possible, for then the judgment which we assert would hold good of itself and might imply contradiction. But if we speak of all propositions of the n^{th} order we assert a proposition of the $(n+1)^{\text{th}}$ order which does not hold good of itself, and we then avoid both a contradiction and a vicious circle.

We have here only skimmed over this delicate and difficult theory of propositional functions (which is, moreover, not yet fixed); but we cannot leave it without calling attention to the close connexion which exists between it and the calculus of probabilities. This classic theory, which owes its development and its celebrity to a few illustrious mathematicians, is in reality a branch of Logic, for it deals essentially with propositions or rather with propositional functions. What do we really mean by *probability*? In the first place, it cannot be concerned, whatever may be said to the contrary, with the probability of an event, for an event is essentially particular and determinate; it happens or does not happen or, in better words, it exists or does not exist, it is real or imaginary; there is no middle place. Probability can be nothing else than the quality of certain *judgments* which we pass upon events in order to predict or conjecture them. But of what judgments? If it is a question of a judgment passed on a particular event it is once again fully determined; it can, therefore, only be true or false. The epithet "*probable*" can only be applied to a judgment which may be true in certain cases and false in certain others. But such a general or indeterminate judgment, which holds good indifferently of any "case" of a series is nothing else than a propositional function. And this is true even of the two extreme cases, *i.e.* of the judgment which is *always true* and the judgment which is *always false* (called respectively *certain* and *impossible* in the calculus of probabilities); for they, too, hold good, not of a unique case, but of a series of cases.

We know how the probability of a judgment is defined

mathematically. Supposing that we had some means of numbering all the "possible cases" (whether this number were finite or infinite); the probability of a judgment is the relation of the number of "favourable cases" to the number of "possible cases"; *i.e.* of the number of cases in which it is true to the total number of cases for which it is defined or which determine it or for which it can have a meaning. Now this definition is based on this hypothesis, that all the possible cases (favourable and unfavourable alike) are *equally possible*, *i.e.* may happen indifferently. And this hypothesis is not always true (this is why we distinguish between subjective and objective probability), and indeed, speaking strictly philosophically, it is never true; it is contrary to determinism, the fundamental postulate of science, to admit that several events are simultaneously possible; there is only one possible logically and physically, and that is the one which actually happens. But in our ignorance (more or less provisional and voluntary) of the sum-total of determining causes, it suits our purpose to consider several events as equally possible, *e.g.* that a die can fall indifferently on any one of its six faces. This hypothesis is doubtful in any case, and, as already said, contrary to the principle of determinism. Nevertheless it has an objective value if the die is homogeneous and symmetrical, while it has none if it is "loaded." What does this mean? Nothing else but this: In the first case, out of a great number of throws, the number of times which each of the different faces will appear will approximate towards equality and will become the less unequal the more the number of experiences is augmented; in the second case, on the contrary, the numbers will remain unequal, and will tend towards different ratios to unity. It is this which constitutes the difference between the probabilities called *a priori* and *a posteriori*. Given an untried die, and no reason to suppose one face will appear more often than another, it will be admitted (subject to fuller information) that the probability is equal for all faces, of which the value for each one is $\frac{1}{6}$. This is *a priori* probability. But if after a great many throws it appears that the relative number of the appearances of 1 is $\frac{9}{60}$ and that of the opposite face 6 is $\frac{11}{60}$, we should say that these fractions measure the real probability of the two faces. Here we have *a posteriori* probabilities. The latter have evidently more objective value than the former, and yet they are not different in nature. Neither the one nor the

other, for instance, permit us to predict which face will appear at the next throw. They only admit of betting fairly (or with equal chance) on a sufficiently large number of throws. What does this mean but that they simply express the relation, real or supposed, of the number of favourable cases to the number of possible cases, *when these numbers are sufficiently high*?

The upshot of the preceding is that every probability is a property of a propositional function; it is as it were its *co-efficient of truth*, which assesses the proportion of cases in which it verifies itself (*i.e. becomes true*). This co-efficient may vary between 0 and 1, and attains its two extreme values in the case of a proposition *always false* or *always true*. This explains the use of these two numerical symbols first for *always true* and *always false* and then for *true* and *false*. For here, as everywhere, the inventing mind has begun with the complex, to mount afterwards to the simple.

The existence of an entire system of logical calculus, based on the consideration of probabilities, *i.e.* the system of MacColl, proves that the calculus of probabilities is a chapter of Logic, especially when we take into consideration the fact that this system arose out of the study of certain problems of probabilities. MacColl considers general propositions only, *i.e.* propositional functions, and he classes and qualifies them according to whether they are certain (value 1), impossible (value 0), or variable (value θ, intermediate between 0 and 1). These expressions are unfortunate; in the first place, because all these propositions are variable, even those which are always true or always false. Secondly, we must guard against confounding the *always true* with certainty and the *always false* with impossibility. Only necessary propositions (*e.g.* mathematical truths) deserve the epithet of *certain*, and only their negations that of *impossible*. The *always true* or *always false* only constitute an *actual*, purely empirical certitude; it is the certitude of throwing 6 with a die of which all the sides bear this number. It has nothing in common with the certitude of necessary truths, which is of quite another order. Many paradoxes and even errors are due to the fact that we seem thus to associate in the calculus of probabilities the certain and the probable as though they were homogeneous. In reality, what we call certitude is only the maximum of probability—that of a lottery in which all the lots draw a prize—but which logically is still a probability.

The analogy between the logical calculus and the calculus of probabilities becomes more manifest if we remark that logical addition and multiplication correspond to arithmetical addition and multiplication. We know that the probability of a composite event is equal to the *product* of the probabilities of the simple events involved. But the distinction between "simple" and "composite" events has no objective basis; it depends entirely upon our manner of conceiving and expressing them. What do we really mean by a "composite event"? We mean a proposition which affirms simultaneously several simple events, and which is the *logical product* of propositions which affirm separately each one of them. And this correlation is based on a combinatory theorem; let there be two propositions A and B, of which one is related to m cases and the other to n cases. If we combine one by one each case of the first with each case of the second, the number of cases thus obtained will be $m \times n$. Now this is precisely the number of cases of the proposition AB, the simultaneous affirmation of A and of B. On the other hand, if m', n' be the cases favourable to A and B respectively, the number of cases favourable to AB will be, in virtue of the same theorem, $m' \times n'$, and we have the arithmetical equation:

$$\frac{m' \times n'}{m \times n} = \frac{m'}{m} \times \frac{n'}{n}$$

which means: the probability of the composite proposition (the logical product) AB is equal to the arithmetical product of the probabilities of the simple propositions A and B. The same is true of logical addition. We know that the probability of an alternative is equal to the sum of the probabilities of the simple events; but here we always suppose that these simple events are unconnected and hence have no case in common:[1] we suppose, too, that the total number of cases possible for the alternants is the same. Let n be this number and m, m', the number of cases favourable to the two alternants A and B, their respective probabilities are $\frac{m}{n}, \frac{m'}{n}$. What is the probability of their alternation, *i.e.* of their *logical sum* A + B? It has the same number of possible cases n. The cases favourable to it

[1] For example, if an urn contain balls of diverse colours, if the probability or drawing a blue ball is $\frac{1}{3}$, and that of drawing a red ball $\frac{1}{4}$, the probability of drawing a blue *or* red ball is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. We suppose, of course, that no ball is at once blue and red; in other words that the probability of drawing a ball which is red *and* blue is 0.

include, of course, all those favourable to A, in number m, in addition to all those favourable to B, in number m'. By hypothesis these two have no case in common, hence the number of their sum is $\frac{m+m'}{n}$. And we have the equation:

$$\frac{m+m'}{n}=\frac{m}{n}+\frac{m'}{n}.$$

"The probability of the alternation is equal to the sum of the probabilities of the simple propositions."

Negation itself is susceptible of a very simple mathematical expression. Let n again be the number of possible cases, and m the number of cases favourable to a proposition A; its probability is $\frac{m}{n}$. What is the probability of its negation A′? The number of cases favourable to A′ is evidently $n-m$, the number of cases unfavourable to A; then its probability is

$$\frac{n-m}{n}=1-\frac{m}{n}.$$

Hence the probability of the negation of A is the difference between 1 and the probability of A.

All that we have been saying explains the origin of Boole's logical calculus—the first complete system which has served as a foundation for more modern ones, and why his operations were modelled on arithmetical operations, even in their symbolic expression. Logical multiplication and addition are analogous, as we have seen, to the arithmetical operations of the same name; and in order to render this analogy more perfect it is necessary to admit that the summands have no common element, which is precisely what Boole did. That is to say, he represented the negation of A by $1-A$, which is sufficient to prove that he conceived his logical calculus on the model of the calculus of probabilities.

III. THE LOGIC OF CONCEPTS.

We are now in a position to attack the theory of concepts, which in ancient Logic occupied the first place, before the theory of propositions, as if the formation of concepts were the

primordial and essential business of the understanding. But that this theory is subordinate to that of propositions is shown by the definition of the concept itself: "*A concept is a propositional function with one variable.*" This paradoxical definition requires some explanation.

Let us consider the proposition: "Paul has only one eye." If for Paul, the name of a person of our acquaintance, we substitute the variable x, we obtain the propositional function, "x has only one eye." Let us suppose that the sum-total of the possible values of x is the sum-total of men. This propositional function would be false for the greater part and true for some of them; the sum-total of these last *values* we will call the *extension* of x. Here we come on a general fact: every function with one variable determines a *class*, which is its extension, *i.e.* the sum-total of the *values* which verify it. To this class a *name* can be and generally is applied; in our example the name is "one-eyed." The function in question then defines the class of one-eyed and constitutes the concept "*one-eyed.*" Similarly, the function "x has lost his wife" defines the concept of *widower*; the function "x has lost (a part of) his hair" defines the concept of *bald*, etc. These names (*one-eyed*, *widower*, *bald*, etc.) signify nothing other than the functions which define them; to say that "x (some particular person) is one-eyed" is to say that "x (some particular person) has only one eye," and so on. In other words, "a one-eyed" is "one (a man) who has only one eye," and similarly with the others. And the extension of the function is identical with the extension of the concept itself. The sum-total of the people "who have only one eye" is identical with the class "one-eyed," etc.

We have in the above tacitly laid down an axiom and a definition; the axiom is: "corresponding to every propositional function is a class which constitutes its extension"; the definition is that of this extension itself. We need a symbol to represent the class which corresponds to a function. Let $\phi(x)$, or simply ϕ, be this function: its extension will be represented by $x \ni \phi$ ($\ni$ = the Greek letter ε reversed), which may be read "x such as ϕ" ("let ... be true" understood). Generally, the symbol $\ni$ can be translated by a relative pronoun (either in the nominative or in any other case) "who has only one eye." The effect of this symbol is to transform a function into a class, or a proposition into a concept; it is thus the natural and

necessary transition between the calculus of propositions and the calculus of concepts.[1]

Logically the concept and the corresponding class are posterior to the proposition which defines them. Nevertheless they can be considered in themselves, independently of their origin, and we shall have to deal with the symbols a, b, c, . . . which represent both the concepts determined and their corresponding classes. We shall then require a symbol to express the fact that a given individual belongs to a given class: *e.g.* that "Paul is one-eyed." This symbol (which translates the copula *is* in those cases where its subject is an individual) is ε. The formula $x \varepsilon a$ reads: "x (an individual) is an a" (belongs to the class of a's). This symbol (the first letter of the word ἐστι is the graphical inverse of the symbol $\ni$, because it is also its logical inverse. It "transforms" a class into a proposition, and it enables us to pass from the calculus of classes to that of propositions. Similarly, the class a can be defined by a function ϕ:

$$a = x \ni \phi(x),$$

and conversely, a function ϕ can be defined by the class a:

$$\phi(x) = x \varepsilon a.$$

"If a is the class of x's which verifies ϕ, to affirm ϕ of x is to say that x is an a." "The axiom by which we have affirmed the existence of a class which is the extension of the function ϕ also takes the form of the *axiom of reducibility*" (Russell).

"Every propositional function with one variable can be reduced to a judgment of predication" (*i.e.* to a judgment which affirms a predicate of a subject or which affirms that the subject belongs to a certain class which is the extension of this predicate). Symbolically, we can say, every propositional function $\phi(x)$ is equivalent to a judgment of the form $x \varepsilon a$. But it will be seen under what condition this axiom can be enunciated, *i.e.* the condition of admitting or of postulating that to every propositional function there exists a corresponding class. And we may go on to ask ourselves whether there is a concept corresponding to every class (as Leibniz boldly maintained). Doubtless this is so if we admit as a concept the notion "which-belongs-to-this-class."

[1] This at the same time explains the logical rôle of the relative pronoun *who*: it transforms an assertion into a quality, a proposition into an epithet: "Caesar conquered Gaul" is an assertion; "Caesar, who conquered Gaul . . ." is only a name accompanied by an epithet, analogous to a simple *adjective*.

At any rate, it is only in this very general sense that we can affirm that every assertion about an object (an individual) is equivalent to attributing to it a predicate, that of a "quality" or a "property." This remark very notably restricts the scope of the logic of concepts, while the ancient logic reduced all judgments to judgments of predication, *i.e.* those which attribute a predicate to a subject.

In addition to the above it is important to note that the judgments of predication which we have arrived at so far are exclusively *singular* judgments, of which the subject is an *individual.* The judgments which the ancient logic considered fundamental are really more complex: we shall now proceed to show how they are translated and interpreted in modern logic.

What is the signification of the universal affirmative proposition: "Every a is b" (where a and b stand for concepts)? It signifies that every individual which belongs to the class a also belongs, in virtue of that fact, to the class b; in other words, that the class a is contained (entirely) within the class b. This is expressed symbolically by the formal implication:

$$(x \,\varepsilon\, a) <_x (x \,\varepsilon\, b);$$

"x is an a" implies that, whatever be the value of x, "x is a b." Or again: "If x is an a, it is a b." For example: "All men are mortal" signifies: "If x is a man, x is mortal." We see that the pronoun *all*, symbol of universality, disappears and is replaced or interpreted by the *generality* of this implication, which is indicated by the symbol x used as an index. In the last resort this generality consists in the indeterminateness of x; its ultimate foundation, then, lies in the notion of the *variable.* We must note carefully that in this implication the variability of x is by no means restricted to the class a; for if x is not an a the hypothesis is false and consequently the implication is verified (for by definition it is equivalent to the alternative: "Either x is not an a or it is not a b"). It follows that a proposition of this form in no wise implies the *existence* of its subject, *i.e.* that there are a's. According to a remark of Mr. Bradley's, "trespassers will be prosecuted," in no way implies that there are or will be trespassers; it only states that *if there are any* they will be prosecuted. If there are no a's the implication will be verified *ipso facto.* The fictitious proverb: "All the carts that go to Crowland have wheels of gold and

silver" means in reality that no cart ever goes to Crowland (a village inaccessible to wheeled traffic), and shows that on this point ordinary common sense is in accord with the opinion of the most rigorous logicians.

Thus to the relation of *implication* between two propositions (whether or not reduced to the predicative form: $x \varepsilon a$, $x \varepsilon b$) corresponds a relation of *inclusion* between their two extensions (the class a is included in the class b). It is, then, natural in the calculus of classes, to represent this relation by the same sign, and to write by definition [1] $a < b$, when we have

$$(x \varepsilon a) <_x (x \varepsilon b).^2$$

To anticipate, the calculus of classes is formally analogous or identical with the calculus of propositional functions: the formulae are the same but are differently interpreted; and this results from the formal analogy between implication and inclusion. We will now briefly show how logical operations are interpreted in the calculus of classes.

The *product* of two classes ($a \times b$) is the sum-total of the elements common to these two classes. It may be defined as follows:

$$a \times b = x \ni [(x \varepsilon a)(x \varepsilon b)];$$

"ab is the sum-total of x's of which one can say at the same time, x is a, x is b." In other words:

$$x \varepsilon ab = (x \varepsilon a)\ (x \varepsilon b).$$

The *sum* of the two classes ($a + b$) is the sum-total of the elements which belong to one or the other. It is defined as follows:

$$a + b = x \ni [(x \varepsilon a) + (x \varepsilon b)];$$

"$a + b$ is the sum-total of the x's, of which it can be said: x is a or x is b." In other words:

$$x \varepsilon (a + b) = (x \varepsilon a) + (x \varepsilon b);$$

"To say that x is 'a or b' is to say that x is a or x is b."

[1] We must notice here that in $a < b$ there is no index to the sign $<$, because there is no longer a variable; it is the question of a relation between two constant objects, a and b.

[2] In reality we began by considering the inclusion of one class in another, and have passed on from that to the implication of propositions (which explains the form of the sign $<$). But this is not the first time that the logical order has reversed the historical.

The *negation* of a class is the sum-total of the individuals which do *not* belong to that class:

$$a' = x \ni (x \varepsilon' a);$$

[$(x \varepsilon' a)$ signifies: x *is not* an a]. In other words:

$$(x \varepsilon a') = (x \varepsilon' a).$$

"x is not-a" signifies "x is not a." We see that the negation of classes is defined by means of the negation of propositions. We have still to find the interpretation of 0 and of 1 in the calculus of classes: 0 is the class which corresponds to the propositions *always false*; 1 is the class which corresponds to the propositions *always true*.

This is expressed as follows:

$$[\phi(x) =_x 0] <_\phi [0 = x \ni \phi(x)];$$

"If a function ϕx is always false, its extension is 0, and that whatever be the value of ϕ." The extension of functions always false is also nought: hence the class *zero* is that which contains no element.

$$[\phi(x) =_x 1] <_\phi [1 = x \ni \phi(x)].$$

"If a function $\phi(x)$ is always true, 1 is its extension, and that whatever be the value of ϕ." The extension of a function always true, then, is the sum-total of the values which can be attributed to its variable, it is the total class which contains all possible individuals. Such is class 1, called by Boole the *universe of discourse*.

In virtue of the analogy between implication and inclusion we get again for 0 and 1 the characteristic properties:

$$0 <_x x, \qquad x <_x 1.$$

The second of these formulae is evident, no matter what class is contained within the total class. The first is not so apparent: "The class nought is contained in any class whatever." It is worth admitting, however, for the sake of analogy and symmetry.

The equality of classes can be defined similarly to that of propositions:

$$(a = b) = (a < b)(b < a).$$

To say that two classes are equal is to say that they reciprocally contain one another: "All a is b, and all b is a." We can deduce from this:

$$(x < 0) = (x = 0), \qquad (1 < x) = (1 = x).$$

What, then, is the signification of the universal negative proposition: "No a is b"? It simply signifies that "all a is not-b," *i.e.* that the class a is (entirely) contained in the class not-b, since the latter contains all which is not b; it is, then, equivalent to a universal affirmative with a negative predicate. It is written:

$$a < b'.$$

"To say that all a is not-b is to say that the class ab is null and that the classes a and b have no common element." This affords a verification of our symbolic translation. It will be noticed that we also have:

$$(ab = 0) = (b < a'),$$

since the first equation is symmetrical in relation to a and b; "All a is not-b," then, is equivalent to "All b is not-a"; or "No a is b" is equivalent to "No b is a." We thus re-discover the rule of *simple conversion* for the universal negative proposition. We see that in this proposition the negation does not apply to the subject (in spite of appearances: "*No*" . . .), but exclusively to the predicate.

We have still to translate particular propositions. The simplest way of doing this is to note that they are respectively the contradictories, *i.e.* the negations of the two universals. The *particular affirmative*: "Some a is b" is the negation of the universal negative: "No a is b." It is translated, then, by "denying" the formula of the latter:

$$a <' b', \text{ or } ab =' 0.$$

It will be noticed here that the negative sign is applied to the *copula* ($<$ or $=$), *i.e.* to the proposition as a whole. Similarly, the *particular negative* is translated by the negation of the universal affirmative:

$$a <' b, \text{ or } ab' =' 0.$$

Thus these particulars deserve to be called negative, while the propositions actually called negative simply have a negative product, which is a detail of no importance (and all the less so since a term is positive or negative by convention or arbitrarily, and since the negation of a negative term is the positive term corresponding to it: $(a')' = a$[1]).

[1] We see that the notorious "quantification of the predicate" has nothing in common with Logistics, for the terms *all* and *some* are precisely what disappear in the symbolical expression of propositions, being translated by a relation between the two terms (classes) considered in their *totality*. It is therefore absurd to consider these

These formulae show the existential import of the four classical propositions, which has given rise to so many discussions. A universal in no way implies the existence of its subject (*i.e.* of the corresponding class), as may be seen from its definition: "All a is b" means "If x is a, x is b"; this does not assume the existence of a, but merely signifies that: "If there are a's they are b's." This is still more evident from the formula $ab' = 0$. It really expresses the nullity (non-existence) of the class ab': "a which is not-b does not exist." But it neither expresses nor implies the existence of any class whatever, either a or b.[1] This is still more evident in the case of the universal negative: "No a is b," which is equivalent to "ab's do not exist." This assertion is manifestly compatible with the nullity of the classes a or b, or even a and b.

The contrary is true of particular propositions owing to the fact that they are the negations of universals. Since the latter deny the existence of certain classes, the former affirm the existence of the same. To say that "Some a is b" is to say: "ab's exist," *i.e.* to affirm the existence of the class ab, and consequently, of the classes a and b. Similarly, to say: "Some a is not-b" is to affirm the existence of the class a-not-b, and hence also of the classes a and not-b. This perfect symmetry or reciprocity is a necessary result of the fact on which the old Logic is based, *i.e.* that the propositions A and O, E and I are mutually contradictory.

But here is an important consequence. If a universal has no existential import, and if a particular has one, it is impossible (that is to say, illogical) to infer a particular from a universal, for from a proposition which affirms no existence (or which denies one) there is no means of deducing a proposition which affirms an existence (short of invoking an existential premiss, as we shall see presently).[2] It results from this that *subalternation*

notations as qualities or attributes of the subject, and to attempt, by means of a purely verbal and illusory analogy, to apply them to the predicate also. This so-called extension or generalization of formal Logic, then, consists simply in making false windows for the sake of symmetry.

[1] Indeed the logical calculus shows that from a formula with a positive copula (an implication or equation) we can never deduce a formula with a negative copula (non-implication or non-equation).

[2] It will be understood that we are speaking always and only of the *existence* of a *class*, *i.e.* of its non-nullity (the class a is not null, *i.e.* it contains some individuals), never of the existence of *individuals*.

is false as well as *conversion per accidens.* From "All a is b," which does not imply the existence of a, we cannot directly infer "Some a is b," which implies the existence of a, nor, consequently, "Some b is a," which is equivalent to the preceding by simple conversion.[1] If we are to make this deduction legitimately, we must admit as a hypothesis the existence of a (or a hypothesis which implies it). Indeed, if we proceed from the premiss:

$$a =' 0, \text{ or } ab + ab' =' 0,$$

and if we assume $ab' = 0$ in virtue of the hypothesis (all a is b) we obtain the result

$$ab =' 0,$$

i.e. "Some a is b," or indifferently, "Some b is a." We shall see the importance of this observation when we come to the theory of the syllogism.

We have just shown that the judgments of the old Logic are in reality complex and derived because they refer to classes and are defined by means of singular judgments, referring to individuals ($x \varepsilon a$). Thus modern Logic not only admits the consideration of individuals, but is founded upon it. But what is an individual? Can an individual be defined?

Schröder has attempted to define "individual" as follows: "An individual is a class which is entirely contained or excluded in relation to any other class" (for, in truth, an ordinary class which contains more than one individual cannot be entirely contained or excluded in reference to any class whatever; if it only contains two individuals, there will be at least one class which will contain the one and exclude the other). Or under another form, "The individual is a class such that for every other class x, it is contained in x or in not-x." But he here defines the individual as a function of the idea of a class, and he really defines it as a "singular class" rather than as an individual. Now we ought to distinguish strictly between the singular class and the individual which constitutes it.[2]

If the individual be undefinable we can at least define the

[1] Simple conversion is indeed valid for the particular affirmative proposition as for the universal negative of which it is the negation, and for the same reason of symmetry.

[2] The fact that one class—even a singular one—is contained in another is represented by the symbol $<$, while the fact that an individual belongs to a class (even singular) is represented by the symbol ε.

identity of two individuals. This can be done as follows ($\equiv$ being the sign for this identity):

$$(x \equiv y) = (\phi x <_\phi \phi y).$$

Definition: "To say that x is identical with y is to say that every propositional function relative to x implies the same propositional function relative to y," or, as we say, "everything which is true of x is also true of y." The inverse implication is, of course, equally true, but since it can be deduced from the preceding formula it need not be enunciated here.[1] We may then also say:

$$(x \equiv y) = (\phi x =_\phi \phi y).$$

Thus two individuals are identical when all that can be asserted of one can be asserted of the other. It is, in brief, Leibniz's *principle of indiscernibles*, but we must bear in mind the *absolute* generality of "all" in our enunciation; it comprises all possible assertions, and not only the "intrinsic denominations." It is abundantly evident that if *everything* which is true of x is also true of y, x and y are indiscernible and only constitute one.

This being granted, we may define the singular class, in contradistinction to Schroder, by means of the identity of individuals:

$$[(a \varepsilon 1)] = [a =' 0][(x \varepsilon a)(y \varepsilon a) < (x \equiv y)].$$

Definition: "a is a singular class" means: "The class a is not null," and if x, y are individuals of a, x and y are identical.[2]

In virtue of the formal analogy between the calculus of classes and that of propositions, all the fundamental formulae of the latter hold good for the former. We will recall them, indicating their new interpretation—

Principle of identity: $a < a$.

"All a is a."

[1] In truth, if $\phi x <_\phi \phi y$, we have also, by contraposition, $\phi' y <_\phi \phi' x$ (the negation of ϕ in relation to y implies the negation of ϕ in relation to x); but ϕ being anything whatever, ϕ' is equivalent to ϕ. Then the converse is true.

[2] Here the symbol 1 represents the number 1, for this number is the generic concept (or the class) of singular classes. To say of a class that it is singular is to say that its cardinal number is 1; see Couturat's *Les Principes des Mathématiques*, ch. ii. § 13). Similarly, the logical zero of classes is identical with the arithmetical zero: to say that a class is null is to say that its cardinal number is 0.

Principle of contradiction: $aa' = 0$,

"Nothing is a and not-a at the same time."

Principle of excluded middle: $a + a' = 1$,

"Everything (*i.e.* each individual) is a or not-a."

Law of commutation:

$$ab = ba; \qquad a + b = b + a.$$

Law of association:

$$(ab)c = a(bc); \qquad (a + b) + c = a + (b + c).$$

Law of distribution:

$$(a + b)c = ac + bc; \qquad ab + c = (a + c)(b + c).$$

Law of tautology:

$$aa = a; \qquad a + a = a.$$

Law of absorption:

$$a + ab = a; \qquad a(a + b) = a.$$

Law of simplification:

$$ab < a; \qquad a < a + b.$$

"All ab is a"; "All a is a or b."

Law of composition:

$$(a < b)(a < c) < (a < bc); \qquad (b < a)(c < a) < (b + c < a).$$

"If every a is b and every a is c, all a is bc."

"If every b is a, and every c is a, every b or c is a."[1]

Lastly, *the principle of the syllogism*:

$$(a < b)(b < c) < (a < c).$$

"If every a is b and every b is c, every a is c."

This time we have the principle of the categorical syllogism, such as was contemplated in the classical Logic. It is the formula of the mode *Barbara*. And since this principle is irreducible to any others (especially to the three "laws of thought") the syllogism itself cannot be justified by other principles, and requires a special principle. This justifies Aristotle and the schoolmen, who recognized as a special axiom the *dictum de omni et nullo* against certain modern logicians, or rather meta-

[1] It will be noticed that in these formulae (and their analogues) the sign $<$ has two meanings: as principal copula it signifies implication; as elementary copula (in brackets) it signifies the inclusion of classes.

physicians, who claim to be able to reduce formal Logic to some one principle.

From the *principle of the syllogism* we can deduce the fifteen valid modes of the classical syllogism, by making in the premisses and the conclusion certain legitimate transformations (such, namely, as are conformable to logical laws, like simple conversion and transposition).[1] But we cannot deduce from it the four modes: *Darapti*, *Felapton*, *Bramantip*, *Fesapo*, because they draw from universal premisses a particular (and hence existential) conclusion, which, as we have shown,[2] is illegitimate. This can be verified directly in each one of the modes; for example, in *Bramantip*. From its premisses:

$$(a < b)(a < c)$$

we conclude, in virtue of the law of composition,

$$a < bc,$$

which does not permit of the elimination of the middle term a. To obtain the so-called conclusion:

$$bc =' 0,$$

"Some b is c," we must suppose that $a =' 0$ (an existential premiss falsely implied in the universal premisses: $a < b$, $a < c$); and then:

$$(a =' 0)(a < bc) < (bc =' 0).$$

[1] We know that simple conversion is indicated by the letter s in the names of the modes (*Cesare*, *Camestres*, etc.), and transformation by the letter k (*Baroko*, *Bokardo*). And indeed, if we work the transposition according to the formula of *Barbara*.

$$(a < b)(b < c) < (a < c),$$

we obtain either $$(b < c)(a <' c) < (a <' b),$$

which is the formula of *Baroko*, or:

$$(a < b)(a <' c) < (b <' c),$$

which is the formula of *Bokardo*. The ancient logicians really proceeded in the same manner, for they employed the *reductio ad absurdum* as follows: "If a syllogism is valid we ought to be able to deduce from one of its premisses and the negation of the conclusion the negation of the other premiss." They thus reduced *Baroko* and *Bokardo* to *Barbara* by an inverse procedure from that which we have just indicated. They applied, without knowing it, the *law of transposition*.

[2] A sign of this invalidity is the presence in the names of the four modes of the letter p, which indicates that they are deduced from *Barbara* by a *partial conversion* (*conversion per accidens*), which is an illegitimate mode of inference. For the same reason the *subaltern* modes which it was thought could be deduced from the five universal modes in order to carry up the number of valid modes to twenty-four (six for each figure) are equally invalid, for they are founded on subalternation, which is an illegitimate form of inference.

Thus all the valid modes of the syllogism can be deduced from or reduced to the formula of *Barbara*, and their enumeration would be of no interest.[1] As to their number, nineteen (or fifteen after correction) as given by Aristotle, it results entirely from the verbal forms to which he restricted himself: for example, he admitted negative predicates (latent in the negative propositions so-called), but he did not admit negative subjects.[2] There resulted from this rules which, though valid in the conditions under which they were enunciated, are as arbitrary as are the conditions themselves; for example, the rule that from two negative propositions no conclusion can be drawn.[3] If we free ourselves from the restrictions founded entirely on the ordinary forms of language we shall find not fifteen, but 8192 valid modes.

Mrs. Ladd-Franklin[4] has shown that all the syllogisms can be reduced to a single "inconsistency," which she calls "antilogism," but which is better known under the names of "inconsistent triad" or "Ladd-Franklin formula":

$$(ab=0)(b'c=0)(ac='0)=0;$$

i.e. "These three propositions cannot be true at the same time." Now this formula can easily be reduced to that of Barbara. By transposition:

$$(ab=0)(b'c=0)<(ac=0);$$

by transforming the equations into implications we get

$$(a<b')(b'<c')<(a<c').$$

This is the formula of Barbara with the exception of the accents.[5]

The theory of the syllogism, indeed, played a part of too exclusive importance in the classical Logic. There are many other forms of valid deduction which depend not upon the principle of the syllogism but upon other principles. Hence it

[1] The distinction of the figures with their special rules becomes still more an absolutely idle complication or subtlety.

[2] And this is the sole reason why the scholastics neglected contraposition as a form of immediate deduction. From "all a is b" they deduced "all not-b is not-a," which, having a negative subject, did not figure among the forms admitted as normal. It was, nevertheless, a legitimate manner of *converting* the universal affirmative and the particular negative, while partial conversion (*conversion per accidens*) was illegitimate.

[3] Similarly, if we admit negative terms, we can have valid syllogisms with four terms, *i.e.* with m and not-m, m being the middle term.

[4] Mrs. Ladd-Franklin, art. "Symbolic Logic," in *Encyclopaedia Americana*, and in Baldwin, *Dictionary of Philosophy*.

[5] Cf. Keynes, *Formal Logic*, § 265 ff.

is a mistake to regard the syllogism as the normal and unique type of all deduction. What is characteristic of the syllogism is the elimination of the middle term between two extremes.[1] But Boole has shown that this process of elimination is very general: he regarded all mediate deduction as the elimination of one or several terms of a problem, and the restriction of the conclusion to two terms seemed to him an arbitrary limitation. There are, however, many other ways besides elimination of deducing from an ensemble of premisses the consequences that are implied. For example, we know how to "resolve" logical equations like algebraic equations and to find out from them the "value" of such and such an "unknown." But that again is a special and one-sided procedure. The most general method of finding (1) all the equivalent forms, (2) all the consequences, (3) all the causes,[2] (4) all the "roots" of any given system of premisses has been invented by Poretsky.[3] It is a purely algebraical and quasi-mechanical method of which we cannot here discuss the principle but which furnishes among many others all the inferences which can be obtained by the other methods. It is a long and somewhat tedious process, for it leaves no scope for the invention or personal ingenuity of the calculator, but it is "exhaustive."

The theory of syllogism is also very much restricted in another direction, for it is based on the respect paid to the four propositional types, A, E, I, O, and these types are neither the most general nor the most simple. They are propositions with two terms; moreover these two terms are concepts or classes;[4]

[1] As a matter of fact, Boole's formula of elimination $(ax+bx'=0)<(ab=0)$, by which we obtain the "resultant" of the elimination of x in the equation of the first member is only another form of the formula of Barbara:

$$(ax=0)(bx'=0)<(ab=0),$$
$$\text{or } (a<x')(x'<b')<(a<b').$$

[2] In Logic we call the *cause* of a proposition every other proposition of which the first is the consequence.

[3] Cf. my *Algèbre de la Logique*, §§ 42-46 and the bibliography.

[4] From the formal point of view, doubtless, singular judgments are identical with universal judgments, and *e.g.* syllogisms about "Socrates" are accepted without scruple. Nevertheless there is an essential difference between the two relations $<$ (the inclusion of two classes) and ε (the belonging of an individual to a class), and consequently between the two syllogistic formulae which are confused:

$$(a<b)(b<c)<(a<c)$$
$$(x\,\varepsilon\,a)(a<b)<(x\,\varepsilon\,b)$$

"x is an a; all a's are b; therefore x is a b."

finally, the relation between these classes is entirely one of inclusion or non-inclusion; the copula *is* cannot even express their equality; there must be two propositions to enunciate the two inverse inclusions.[1] By limiting itself to the consideration of this one kind of terms and to this one kind of relations, classical Logic forgot that thought takes for its object all sorts of *terms*, of which the most simple are individuals (concrete objects), and that it perceives or establishes between them all sorts of diverse and heterogeneous relations. And this *logic of relations* is one of the principal conquests of modern Logic.

IV. THE LOGIC OF RELATIONS.

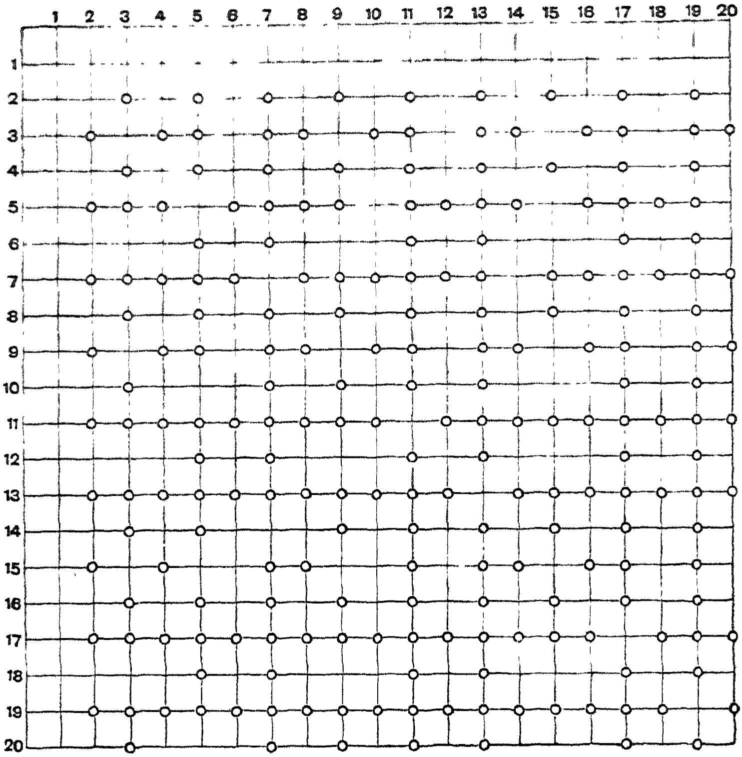
To understand the importance of the logic of relations it is sufficient to notice that the greater number of ordinary propositions express relations. There are no doubt propositions without terms (impersonal verbs), *e.g.* "it rains," "it thunders," and also propositions with one term, or judgments of predication (intransitive verbs), *e.g.* "the sun shines," "the child sleeps." But the greater number of ordinary propositions contain two or more terms (transitive verbs), *e.g.* "Paul loves Jean," "Peter gives a book to James," etc. Now all these express relations between various terms (Paul and Jean, Peter, James and the book). What, then, is a relation? It is a proposition having several terms; or more exactly, since we ought to consider these terms as variables (empty places), which are not part of the relation, *it is a propositional function having several variables.* In *Paul loves Jean* we can replace Paul and Jean by any other terms: that which remains the same is the relation *loves*, or, more exactly, the propositional function: "x loves y," for "*loves*" has no sense unless it is accompanied by a subject and an object. Similarly, the verb *give* has no sense unless it has three terms: the one who gives, the one to whom something is given, and what is given.

Like all functions, a relation can be represented by symbols analogous to: $\phi(x, y, z)$ when x, y, z represent terms or variables. But for binary relations (*i.e.* relations having two terms) the more easily read symbol xRy is preferable, when x, y represent the terms and R the relation.

[1] The so-called *toto-total* of Hamilton: "All a is all b," is nothing more than the synthesis of two propositions: "All a is b; all b is a." Symbolically:

$$(a=b)=(a<b)(b<a).$$

As with propositional functions, a relation is true for certain systems of values of variables and false for others. The sum-total of the first constitutes its extension. The extension of a binary relation may be represented as follows: take a sheet of paper ruled for double entry; arrange in each one of the entries (by preference in the same order) all the terms for which the relation can be defined (*i.e.* has a sense), one entry corresponding



to the value of *one* variable (the first), the other to the value of the other (the second). To each system of values having two variables there is a corresponding point on the table, the intersection of the lines corresponding to these values (as in the table of Pythagoras). Mark a black point for every system of values which verifies the relation: the ensemble of the points so marked (the figure which they make) represents the *extension* of the relation. It is what is called the *matrix*. We will give as an example of a matrix that of the relation "prime to each other" of the first twenty whole numbers.

A scheme of this kind is extremely useful: in the first place it is synoptic; the figure given above resumes and condenses 400 judgments on the twenty numbers combined in pairs; then it represents at sight the disposition of a relation in a given ensemble, and permits of our grasping its qualities almost at sight (with a little exercise). Thus the matrix given above is symmetrical with regard to its principal diagonal,[1] which shows that the relation itself is "symmetrical." The principal diagonal of the matrix is empty; this shows that the relation is incompatible with identity (does not take place between a term and this term itself); and so on.

We understand, thenceforth, that we can study the matrices instead of the relations themselves, since all the properties of the latter are reflected in the former. This method, in short, consists of replacing the relations by their *extension*. It is that of Schröder. Other logicians prefer to consider the relations in themselves, in their *intension*. The first method is more convenient for the calculus and formal treatment, the second deals more with the meaning and is better suited for verbal interpretation. It is the one which we prefer to use here.

Relations (like classes) are capable of two fundamental "relations," *i.e. equality* and *inclusion*.

Inclusion can be defined by means of implication: We say that a relation R_1 is contained in a relation R_2 if each time that it occurs in a system of values the relation R_2 occurs for the same system. This is written (the inclusion of relations being again represented by the sign $<$):[2]

$$(R_1 < R_2) = (xR_1y <_{x,\,y} xR_2y).$$

Definition: Thus the inclusion of relations is equivalent to their implication: to say that R_1 is contained in R_2 is the same as to say that it *implies* it.

The equality of two relations is defined by means of two inclusions: the relations R_1, R_2 are said to be *equal* if they mutually contain one another.

$$(R_1 = R_2) = (R_1 < R_2)(R_2 < R_1).$$

[1] The principal diagonal is that which starts from the upper left-hand corner; it is the symmetrical axis of the two entries.

[2] It must be noted that while the definition is applicable to any relations, the formula by which we translate it assumes that the relation is binary.

Definition: The logical sum and product of two relations can be defined by the following formulae:

$$x(R_1 \times R_2)y =_{x,y} (xR_1y)(xR_2y).$$

Definition: "To say that the relation $R_1 \times R_2$ exists between any two terms x, y, is to say that each of them exists between the said x and y." [1]

$$x(R_1 + R_2)y =_{x,y} (xR_1y) + (xR_2y).$$

Definition: "To say that the relation R_1 and R_2 exists between any two terms x, y, is to say that one or the other exists between the said x and y." [2]

The negation of relations may also be defined:

$$xR'y =_{x,y} (xRy)'.$$

Definition: "To say that the relation *not-R* exists between any two terms x, y, is to say that the relation R does not exist between the two terms." [3]

Thus the three logical operations apply to relations as they do to propositions and to classes, and all the formulae of the logical calculus hold good for them also. But in addition they are susceptible, as relations, of special operations.

First of all, *conversion*, an operation analogous to negation, in that it transforms one relation into another. The converse of a relation (R^c or converted R) is the relation which exists between the same terms after they have been inverted:

$$xR^cy =_{x,y} yRx.$$

Definition: "To say that the relation R^c exists between x and y (in this order) is to say that the relation R exists between y and x (in this order)." [4]

Conversion has properties analogous with those of negation. It is uniform, *i.e.* the converse of a given relation is unique. And the converse of the converse reproduces the original relation:

[1] *e.g.* "x is the father *and* master of y" = "x is father of y, and x is the master of y."

[2] *e.g.* "x is the father *or* master of y" = "x is the father of y or x is the master of y."

[3] The relation *not-R* is true in all the cases in which R is false, and false in all the cases where R is true. Its matrix is deduced from that of R by substituting empty places for all the points and points for all the empty places.

[4] *e.g.* the converse of *larger* is *smaller*; the converse of parent (father or mother) is child (son or daughter), etc. The effect of conversion is to turn the matrix round its principal diagonal.

$(\breve{\breve{R}}) = R$ (the *law of double conversion* analogous to the law of double negation).

But the most important operation in the calculus of relations is that of *relative multiplication* (which must not be confounded with logical multiplication). If there exist together two relations, R_1, R_2, such that xR_1y, yR_2z, *i.e.* that the second term of the first is the first term of the second, there exists between the extreme terms x, z, a relation R_3, which is the relative product of R_1 and R_2, and which is represented by $R_1 * R_2$.

$$x(R_1 * R_2)z =_{x,z} (xR_1y) \times (yR_2z).$$

Definition: This logical operation is well known: the father of the father is the grandfather; the brother of the father is the uncle; the son of the uncle is the cousin; etc. It will be noticed that relative multiplication (superposition of relatives) is generally translated in speech by the preposition *of*, and that the most frequently recurring relative products have received special names.

Relative multiplication is not commutative; thus the *father* of the *brother* is not the uncle, but the father or step-father; the *uncle* of the *son* is not the cousin, but the brother or brother-in-law, etc. Similarly, the *law of tautology* does not hold good for relative multiplication; the father of the father is not the father, but the grandfather. This operation, therefore, generates *powers*:

$$R * R = R^2;\ R^2 * R = R^3,\ \text{and so on.}$$[1]

We need not define *relative addition*, an operation to some degree symmetrical with relative multiplication, for it does not correspond with any intuitive and usual combination, and plays a purely algorithmic rôle. Relations are distinguished by the properties they possess with regard to the operations defined above. For example:

A relation R is said to be symmetrical if, directly it arises between two terms x and y, it also arises between y and x (the inverse order); this is written:

$$xRy <_{x,y} yRx.$$

We have evidently at the same time:

$$yRx <_{x,y} xRy,$$

and consequently:

$$xRy =_{x,y} yRx.$$

[1] It is impossible to make a mistake as to the meaning of these powers, for logical multiplication does not engender powers: $R \times R = R$ (father *and* father = father).

Then, in virtue of the definition of conversion,

$$yRx =_{x,y} xR^c y.$$

And the condition enunciated is equivalent to:

$$xRy =_{x,y} xR^c y, \quad \text{or} \quad R = R^c.$$

Thus a symmetrical relation is a relation which is equal to its converse.[1]

A relation R is said to be *transitive* if, as soon as it arises between x and y, and between y and z, it also arises between x and z (it passes from x to z through the intermediate y); this is written:

$$(xRy)(yRz) <_{x,z} xRz.$$

Now, $$xRy \times yRz =_{x,z} xR^2z,$$

therefore, $$xR^2z <_{x,z} xRz, \quad \text{or} \quad R^2 < R.$$

Thus the transitive relation is a relation of which the second power (the square) is contained in (or implies) the relation itself.

The relation of *equality* is symmetrical and transitive; the relation of *non-equality* is symmetrical but not transitive; the relations *larger than*, *smaller than* are transitive but not symmetrical (they are the converse (Fr. *converse*) of one another). The relations father, son, uncle, nephew are neither symmetrical nor transitive. The relation *brother* (or *sister*) is symmetrical and transitive. The same would be true of the relation *friend* if we could believe the saying, "The friends of our friends are in their turn our friends." But the relation *to love* is not symmetrical; it is possible to love a person without being loved in return; and so on. We see of what "real" importance are these formal properties of the diverse relations. And it is in virtue of these properties that they can be handled in the logical calculus.

There are two relations which play an exceptional part in this calculus, analogous to that played by 0 and by 1 in the calculus of propositions and of classes; they are those of *identity* and of *diversity* (or better, otherness, Fr. *alterité*). *Identity* is the relation which exists between every term and itself, and in no other case. If we indicate it by I, we have:

$$(xIy) = (x \equiv y), \quad \text{or} \quad I = (\equiv).$$

[1] It results from this that the matrix does not change when it is revolved round its principal diagonal; that is to say, it is itself symmetrical in respect of this diagonal.

Or expressed differently:

$$(x \equiv y) < (xIy = 1), \ (x \equiv 'y) < (xIy = 0).^{1}$$

This relation is the modulus of relative multiplication (as 1 is the modulus of arithmetical and logical multiplication), that is to say, it does not change a relative product when it enters as a factor. Thus:

$$R * I = R, \text{ and } I * R = R,$$

for

$$xRy \times yIy = xRy,$$
$$xIx \times xRy = xRy.$$

The negation of identity is *diversity* or *otherness*, represented by I′. These two relations are symmetrical, and hence equal to their converse.[2] Identity is transitive, but otherness is not so.[3]

A relation R is *reflexive* when it exists between any term whatever and itself; that is to say, if we have:

$$xRx =_x 1.$$

This condition may be expressed as follows:

$$xIy <_{x,y} xRy.$$

"If x is identical with y, x stands to y in the relation R."

Thus it is equivalent to: $I < R$, and we see that a reflexive relation is a relation which contains identity (or is implied by it).[4] For example, mathematical and logical equality is a reflexive relation. But it does not coincide with identity, for it can exist between terms which differ from one another (*i.e.* are non-identical). It "contains" identity, but goes beyond it.

A symmetrical and transitive relation is also reflexive if its extension is not null. Indeed if we have: xRy, the relation being symmetrical, we have also yRx. And since the relation is transitive, we can from xRy and yRx deduce xRx, that is to say, it is reflexive.

A relation is said to be *uniform* when to each antecedent there corresponds a single consequent (but a single consequent

[1] The matrix of this relation contains all the points of the principal diagonal and no others.

[2] The matrix of I' contains all the points on the table, except those of the principal diagonal, which are all empty.

[3] From the knowledge that x is other than y and y other than z, we cannot conclude that x is other than z, for z might be x.

[4] This will be admitted at sight, for its matrix contains all the points of the principal diagonal.

may correspond to several antecedents). A relation is said to be *co-uniform* when to each consequent there corresponds a single antecedent (but a single antecedent may correspond to several consequents). The converse of a uniform relation is a co-uniform relation. Finally, a relation is said to be *bi-uniform* when it is at once uniform and co-uniform; when to each antecedent there corresponds a single consequent and to each consequent a single antecedent. Such a relation is evidently symmetrical (equal to its converse). Its terms may be divided in couples, arranged in such a manner that each one of them appears in a single couple only, either as antecedent or consequent.

If R, S are two uniform relations, their relative product is a uniform relation.

If R, S are two co-uniform relations, their relative product is a co-uniform relation.

If R, S are two bi-uniform relations, their relative product is a bi-uniform relation.

We can now establish an important proposition of which the following is the enunciation:

"If R is a uniform relation, the relative product of R and of its converse (Fr. *converse*) is a symmetrical and transitive relation."

The converse (Fr. *réciproque*) of this theorem is still more important; here is its enunciation:

"Every symmetrical and transitive relation (of which the extension is not null) is equal to the relative product of a uniform relation and its converse (Fr. *converse*)."

It is called the *principle of abstraction* because in virtue of this law all symmetrical and transitive relations may be considered as a kind of equation, in this manner,—every time that in an ensemble of objects a symmetrical and transitive relation has been defined which exists between some among them, we conceive, by "abstraction," that the latter have a common element, which is often considered as a magnitude; and the said relation is then conceived as the *identity* of this element in its different objects. For example, if we define the thermic equilibrium of two bodies, and if we find that this is a symmetrical and transitive relation, we can conceive the temperature as a state common to these bodies, and the thermic equilibrium as an equation of temperature. We admit, in reality, a symmetrical

and transitive relation R between the objects x, y, z; we imagine an entity t to which they each stand in the uniform relation S: xSt, ySt, zSt. Consequently, the relation between any two of these is composed of the relation S and of its converse.

$$xRy = xSt \times ySt = xSt \times tS^c y.$$

This is expressed when we say that R is the relative product of S and S^c.

This *principle of abstraction* is very frequently applied in the mathematical and physical sciences, in order to define certain "magnitudes" not directly measurable: the mass, the electric potential, etc. In a word, it permits us to reduce every symmetrical and transitive relation to the equation of an abstract element (often itself imperceptible). It is thus that the *parallelism* of straight lines becomes, in geometry, the identity of their direction (or of their point at infinity); the equivalence of vectors can be reduced to identity of length, of sense and direction, etc. This explains how it is that the relation of equality is preponderant and almost the only one in Mathematics. Every symmetrical and transitive relation is there reduced to some kind of equation, *i.e.* to a partial and abstract identity.

This, again, explains the frequency in Mathematics of *implicit* definitions or definitions *by abstraction.* Here is the type: given a symmetrical and transitive relation R, a function ϕ is defined by saying: "It is assumed that $\phi a = \phi b$ by convention or by definition whenever we have aRb."

This form of definition has been criticized, and rightly, as not being a veritable definition, for this convention of language in no way shows what the function ϕ is (it is not defined in itself, only in its equality) or even if it exists.

Partly to evade this reproach, it has been proposed to say: "The entity ϕ thus defined exists; it is nothing other than that class of things which is equal to one among themselves." For example, *cardinal number* is defined as the common element of "equivalent" classes.[1] If it is then asked: What is a cardinal number itself? the answer is: It is the class of classes equivalent to any one among themselves. For example, the number three is the class of "triads," all equivalent among themselves, and so

[1] Two classes are said to be *equivalent* when a bi-uniform relation can be established between all their elements.

on. It is difficult not to see in this method a kind of vicious circle. For in what are the triads equivalent if it is not because each one of them bears the number three? And is it necessary that there should be several triads in order to be able to extract from them by abstraction the idea of the number three?

However this may be, definitions by abstraction are evidently very imperfect or incomplete definitions, for they do not determine the object defined nor do they in any way prove its existence. Also the frequent use (not to say abuse) which has been made of them in Mathematics (while conducing to the belief that entities can be created simply by a verbal convention) has contributed no little to spread first among mathematicians and then among philosophers the kind of nominalism which exaggerates the part played by the conventional and arbitrary, and indirectly favours those sceptical tendencies fashionable to-day under the name of *pragmatism*.

We have already had occasion to notice the analogy which exists between concepts and relations, which indeed results from their definitions (as propositional functions). Now just as a concept can be reduced in the logical calculus to the class which constitutes its extension, so a (binary) relation can be reduced to its extension, which is the ensemble of the couples which verify it, for a relation implies another relation when the extension of the first is contained in that of the second, and two relations are equal when their extensions are identical. On this point we may remark that the calculus of relations re-enters into that of classes, since the classes of couples are, in short, nothing but a species of classes. This is the point of view from which Schröder treated the calculus of relations; we will not stop to discuss it further here, but will proceed to show the transition by which we pass from relations to concepts.

A relation is a propositional function having two (or n) variables; a concept is a propositional function having one variable. We ought, therefore, to be able to transform a relation into a concept by suppressing one or $n-1$ variables. This is indeed what takes place. Let the relation be: "x is the father of y." If for the variable y an indeterminate term is substituted we obtain: "x is father of someone" or simply "x is father." By reduction to the only variable x, the relation "father of" has become the concept "father." This is the origin of many concepts, which might be called "*relative concepts*." In treating

these concepts their origin must not be forgotten, or we shall confound them with absolute concepts, which express the qualities or intrinsic attributes of the object: this is the source of certain Greek sophisms (for example, from "x is a father and possesses a dog" it is deduced that x is the father of a dog). Such concepts must always be considered as incomplete, or as containing a latent variable, and for "father" we must substitute mentally: "father of some one." Moreover, this transformation of a relation into a concept is not always possible, and it sometimes yields an insignificant result. For example, we cannot say with any sense: "x is equal" or "x is similar." Common sense would immediately ask: "to what?"

We see too how relations give birth to logical functions (non-propositional). Just as the proposition: "x has only one eye" defines the concept of one-eyed (*borgne*), so the proposition: "x begot y" defines the concept of *father*. In reality, it expresses a relation between x and y: but this relation is transformed into a function, "father of y," which becomes an attribute or a property of x. Similarly, "x and y are born of the same parents" defines the relation of fraternity which can be transformed into the function: *brother of*; "x is the brother of y." Now "brother of y" (where y is determined) is a concept and defines a class, the ensemble of brothers of y. Thus the relation: "x-is-brother-of-y" is translated by a judgment of predication: "x is a brother of y," *i.e.* belongs to the class "brother of y"; for the formula xRy the formula $x \varepsilon (\phi y)$ is substituted. Such is the artifice (founded on a form of speech) by which the classical logic attempted to reduce judgments of relation to judgments of predication.[1]

V. METHODOLOGY.

Pascal regretted that we cannot define and demonstrate everything, and regarded this impossibility as an infirmity of the human mind. Modern logicians are more modest, or more reasonable. They do not aspire to defining everything and proving everything; they are content with the reduction to the minimum, complete enumeration and exact formulation of all the indefinable notions and the indemonstrable propositions.

[1] It is by an analogous proceeding in mathematics that an implicit function is transformed into an explicit function; from a "relation" $\Phi(x, y)=0$ we extract (when possible) a "function"; $x=\phi(y)$, or $y=\phi(x)$.

And this in itself is no easy task, without attempting to pursue the chimerical ideal of Pascal.

Definition and *demonstration* are two proceedings analogous to reduction, the one of notions to indefinable notions, the other of propositions to indemonstrable propositions. And it is very necessary that this process of reduction should stop at certain elements primary, or considered as such, for if, by any chance, it could be applied to all without exception we should have a vicious circle either *in definiendo* or *in demonstrando.* We will now proceed to give in their order the rules of these two operations.

If we are to believe the writers on Logistics a *definition* is a logical equation of which the first member is the symbol to be defined, while the second member is composed of symbols already defined or admitted to be indefinable.[1] This conception is exact but too nominalistic; for it considers the two members of the definition as symbols or as a combination of symbols which conduces to the view that logical reasoning (and all deductive science) is a pure play of symbols manipulated and interchanged according to conventional rules. What we ought to say is that the second member *has a meaning*, represents a notion more or less complex, and that the effect of the definition is to attribute this meaning to the symbol which constitutes the first member (and which, by hypothesis, does not yet possess one). A definition, then, is at the least the attribution of a meaning to a symbol or the imposition of a name upon a concept. But this formal and conventional act is not the most essential one: the important part of a definition is the composition and constitution of the concept represented by the second member; the imposition of a name is only an accessory. A definition, then, is essentially the construction of a notion. It will be understood that only those notions are defined which are practically useful, *i.e.* which figure frequently in reasonings. But from the nominalist point of view a new symbol is only adopted to represent in an abbreviated form an ensemble of symbols which, recurring frequently, would become embarrassing and tiresome without one. This means, from the conceptualist point of view, that the notion to be defined forms a stable and permanent

[1] We will call the first member the definiend (Fr. *défini*) and the second the definant (Fr. *définissant*) to distinguish them better from *definition*, which is the equation of both.

combination which occurs sufficiently frequently in reasonings to give rise to the wish to give it a name. Thus logical definitions, from the formal point of view, are indeed definitions of *names*, but they are also, at the same time, definitions of *ideas*. For the rest, it matters very little if it is a question of an idea antecedently formed which is being analysed, or of a new idea which is constituted by the definition itself. This is a psychological or historical consideration which in no way affects the character of the logical operation. This consists always and definitively of two acts; a synthesis which creates the notion, and an analysis which defines it by reducing it to other notions already known.

We have said that a definition is a logical equation, but this is not altogether exact. A logical equation, as we have defined it, is a relation between two terms or members, each of which has its own signification, while a definition is an equation which is assumed between a term which has no meaning and a member which has one, in order to confer upon the first the meaning of the second. An equation, strictly speaking, is a *proposition* and may be true or false; but a definition is a *convention* which is neither true nor false.[1] Without insisting upon the arbitrary character of definition which has often been exaggerated (for it is very much restricted by methodological and practical considerations), we may say that definitions are free (in so far as they are *nominal* and do not pretend to give the sense of a word already known or of an idea already admitted) and consequently, from the logical point of view, not open to dispute. But the convention once "posited" it becomes equivalent to a logical equation, that is to say, we must consider and treat its two members as equal. A definition is not a truth and yet, once admitted, it must be regarded as a truth, for it would be self-contradictory to deny or change it.

It is, then, as logical equations that definitions appear and function in reasonings and, however different they may be from equations by nature and origin, nothing distinguishes them from these as premisses or hypotheses. For what is the characteristic

[1] From the formal point of view, if the definition were a true equation we should be guilty of a vicious circle in defining the equation, *i.e.* in writing: $(a=b)=\ldots$; for that would be to define the sign $=$ by means of the sign $=$. But in reality this sign is equivocal here: the sign of a definition ought to be distinguished from the sign of an equation, for example like this: $=$Df.; and it ought to be enunciated (and is, in fact, so enunciated) "equal by definition."

mark of an equation in deduction? It is the possibility of substituting one of its members for another in another proposition. This is also the rôle and the use of definitions: they permit us to substitute either the definiend (*défini*) for the definant (*définissant*) or the definant for the definiend. In general terms, if we want to prove a proposition concerning a certain term we must substitute its definant in order to analyse it and make explicit the content, but, after various permitted transformations, we re-constitute the definant (*définissant*) (a combination more or less complex) and we finally substitute for it the definiend in order to obtain the proposition which was to be demonstrated. Definition, then, is an essential element of demonstration: for, indeed, how can we hope to establish anything whatever about a term, if not by taking account of its meaning, of its conceptual content? Now this content is furnished by the definition and by it alone, for it is forbidden to attribute to a notion any element which does not figure in its definition, *i.e.* in its construction.[1]

From the above we get the following consequence, paradoxical, but yet evident, *i.e.* that all the propositions of a theory refer, in the last instance, to the indefinable notions upon which it was built up. And, indeed, all other notions are defined, in their last analysis, by means of these notions, which are the only indefinable ones; hence, if the precept of substituting for every definiend its definant were applied regressively to all propositions, the enunciations of propositions (prodigiously complicated as they would become if this were done) would contain nothing except indefinable notions.

A definition usually has a concept as its object, in which case we must guard against the belief that it implies the existence of the corresponding class, *i.e.* the (logical) existence of an object corresponding to the concept. In order to evoke this existence we must either prove it or posit it explicitly. And since every theorem of existence is demonstrated in the last resort by another assumed existence, all existential judgments amount, in the last instance, to postulates of existence.

A definition may also have for its object an individual, but an individual defined by means of general terms, *i.e.* as a class;

[1] From this springs the classical methodical principle: substitute everywhere the definant for the definiend or, as it is usually expressed, replace each term by its definition.

there may, therefore, be several individuals answering to the same definition. We must then prove (or postulate) not only the existence of the class, but also the uniqueness of the defined individual; in other words, we must prove that the class defined contains *at least one* individual, and also that it contains *at most one*. This last demonstration is generally effected by proving that, if two individuals respond to the definition, they are necessarily *identical*.[1] Moreover, it is thus that we prove that a class (the defined class) is *singular*.

It now remains to ask: What is demonstration? It consists in deducing from given premisses or hypotheses the consequences or conclusions which they formally imply in virtue of the laws of Logic. From the algorithmical point of view it consists in passing from premisses to conclusions by means of transformations permitted by the laws of the calculus. There can be no logical and correct demonstration except at this price; we must not take a single step which is not justified by the logical laws: all recourse to "evidence" or to "intuition" must be rigorously excluded. And this rule holds good equally of mathematical demonstration, which is the same as logical demonstration.

It is important to note that the logical laws which are involved at each transition or transformation do not play the part of premisses, and must not figure as such in the calculus. They furnish the formula or type for every elementary deduction, but they must not appear in them as a constitutive element. The principle of syllogism, for example, is the type of all syllogisms, but it cannot be the premiss of any, otherwise such syllogisms would have three premisses; in that case we should want another formula to justify these three-premissed syllogisms, and this formula would constitute a fourth premiss, and so on to infinity.

Amongst the premisses of the deduction there generally figure, as we have said, the definitions of the principal notions which form the subject of the argument, and also propositions relative to the subject-matter of the theory (mathematical, physical, economical, etc.). If, then, the laws of Logic ever play the part of premisses, it can only be in demonstrations of logical science; and the fact that the same laws figure in it sometimes as premisses and sometimes as rules of deduction, explains, without justifying, the confusion to which we have called attention.

[1] See in chap. iii. the definition of the identity of individuals.

There is then no single type of reasoning, as the syllogism, according to the traditional and deeply rooted conception, has been supposed to be. There are as many types of simple deduction as there are of logical laws; and there are in addition innumerable complex types of deduction, of which each one is a combination of simple types. There is no demonstration, however simple it may be, which does not make appeal to several logical laws. When the demonstrations are correct, the truth of the conclusions depends entirely on the truth of the premisses; it has the same value and the same certainty. Now if we go back from premiss to premiss, we shall finally arrive at a certain number of *primary propositions*, admitted as axioms and not demonstrated. These are *true axioms* for the particular theory under consideration.[1] It is in them, then, that the truth of the whole theory is contained; the demonstration only transmits it, without diminution, to all the consequences that can be logically deduced from them. There is then, properly speaking, no *logical truth*, unless it be that mediate and relative truth which is contained in the bond between the conclusions and the premisses; Logic only guarantees the necessary consequences, but it is neither the source nor the judge of the truth of the premisses which are taken as fundamental; and the criterion of their truth and with this of the truth of the whole theory based upon them is essentially extra-logical. The belief that logical deduction establishes any truths whatever, such, for instance, as mathematical truths, is a very widespread error. If such truths present (wrongly or rightly) an appearance of evidence and certainty, they owe this exclusively to the axioms from which they have been deduced (always assuming that the deduction is purely and rigorously logical).

The fundamental propositions of a deductive theory generally enunciate the properties or relations of the fundamental notions, and it must necessarily be so, for otherwise it would not be possible to say anything about these fundamental notions, or consequently to prove anything whatever about them. Now, as we have seen, the real subject-matter of all the "theorems" is the fundamental notions; they do but develop the properties enunciated in the fundamental propositions.

[1] They are thus called in opposition to the *common axioms* which are the principles of Logic. They concern the *subject-matter* of the deductions, while the logical laws regulate its *form*.

Thus all the scientific and objective contents of a theory are contained within its first notions, and in its axioms or postulates. Hence it has been supposed that these latter are to be regarded as *defining* the fundamental notions, since they cannot be formally defined (*i.e.* by nominal definitions); and it is in this sense that we sometimes speak of *definition by postulates*. But this expression is incorrect, for this kind of "definition" is really applied to notions *which are indefinable*.

There is one very important fact on which we must be in no doubt, and that is that for any given deductive theory there is not *any one* system of fundamental notions nor *any one* system of fundamental propositions; there are generally several equally possible, *i.e.* from which it is equally possible to deduce correctly all the theorems. The two systems, of course, depend on one another, and if we change the fundamental notions, or even one of them, we shall be obliged to change the axioms, for these are relative to the notions. Hence the choice of this double system is not arbitrary but to a certain extent free and optional: from the logical point of view it is a matter of indifference, and if one system appears preferable to another it is either on account of reasons of convenience and facility or for quasi-aesthetic reasons of order that we will discuss later. This fact is very important, because it shows that there are in themselves no *undefinable* notions nor *indemonstrable* propositions; they are only so relatively to a certain adopted order, and they cease (at any rate partly) to be such if another order is adopted. This destroys the traditional conception of *fundamental ideas* and *fundamental truths*, fundamental, that is to say, absolutely and essentially.

It also diminishes the metaphysical or epistemological importance that we are sometimes tempted to attribute to one particular system of data above all others. No doubt there must always be a system, but a system is not determined in the nature of things, and depends (partly) on the choice of the demonstrator. We may certainly say that a theory in its ensemble is true (in any domain of reality), but we cannot say that its truth resides essentially in such and such axioms to the exclusion of all others, because it depends on us whether any particular axiom be taken as a theorem and any particular theorem as an axiom, according to the order which we adopt in our deductions. The truth of the ensemble is indeed something

objective; but the deductive order, according to which this proposition is an axiom and that a theorem, is something subjective which depends on our methods and our procedure, on our preference and our convenience, and which seems almost an illustration of the infirmity of the human mind. One is tempted to believe that a more powerful mind would see intuitively the simultaneous truth of the entire theory, without distinguishing axioms and theorems, principles and consequences, merely by recognising the reciprocal interlacement of all the propositions. It seems as though the objective and whole truth had the form of a vicious circle, or of a complex net which has no end and in which everything mutually implies everything else. It is our discursive reason which, imposing on partial truths a linear and successive concatenation, breaks the circle or the net and imposes upon them, more or less arbitrarily, a beginning or a point of departure. This explains how we can (to a certain extent) start from any point we like and take any propositions we like as axioms or postulates.

Nevertheless, as we have said, we are guided in our choice of fundamental data by quasi-aesthetic reasons. Let us see in what these consist. For our fundamental notions, it is natural to prefer a system where these are all *independent of one another*, *i.e.* where no one can be defined as a function of the others; for if a notion can be defined by means of others it is no longer necessary as a primary notion, and can be suppressed. The ideal evidently is to reduce to a minimum the number of primary notions and, other things being equal, that system is to be preferred in which this number is the smallest.

How can we prove that the primary notions are all independent of one another, or that they form an irreducible system? It is not sufficient to allege that we have not been able to define any one of them as a function of the others, or that we do not see the means of doing so; for that proves nothing. This is the method we must employ. The primary notions, not being defined, and only being characterised by the axioms or postulates which bind them together, can receive all sorts of interpretations as long as the latter are compatible with these axioms, *i.e.* verify them. This being so, we prove that a primary notion is independent of the others if we can find two interpretations (both compatible with the system of axioms) which only differ in the meaning attributed to this primary

notion: for then it will be proved that the meaning of this notion can not be deduced from the meaning of the others (by means of the axioms which unite them). This demonstration must be repeated separately for each one of the primary notions, and if we succeed in effecting it for all we shall have established the irreducibility of the system.

Similarly, it is evidently desirable that the system of primary propositions should be irreducible, *i.e.* that no one of them should be deducible from the others, for otherwise we should be able to demonstrate it and hence to suppress it. Here, too, it is not sufficient to say that we have not been able to deduce one axiom from the others, or that we do not see the means of doing so: that proves nothing. There is only one known means of proving that an axiom is independent of all the others: it is to find an interpretation (from the system of primary notions) such that it verifies all the axioms except this particular one. For then it will be proved that this axiom does not follow as a necessary consequence from the others. This demonstration ought to be repeated separately for each one of the axioms, and if we succeed in effecting it for all, we shall have established the irreducibility of the system of axioms.

There is another quality in a system of axioms which we often neglect to verify but which is essential, *i.e.* the consistency or compatibility of the axioms among themselves (this is often called "non-contradiction," because if the axioms were inconsistent the negation of one of them would be implied by the others). Here again there is only one way to prove this consistency, *i.e.* to find an interpretation (of the primary notions) which verifies *all* the axioms.[1]

We cannot deny that, from the philosophical point of view, the methods employed to demonstrate the consistency and the irreducibility of the primary axioms and notions are frankly empirical in character and appeal to intuition. We find an interpretation, *i.e.* an ensemble of *objects* which can be subsumed under the primary notions, and we confirm, by intuition and experience, the fact that they verify such and such axioms. We submit these objects, whatever they may be, to a kind of ideal or imaginary experimentation. Moreover, we admit their

[1] It is in reality the same procedure as that for demonstrating the irreducibility of an axiom to others; what we prove in this latter case is the *consistency* of these other axioms with the negation of the axiom in question.

"existence" non-logical but real), and we assume that these existing objects cannot be contradictory or illogical. But who will answer for it, that these imaginary beings are not "impossible" in some latent manner, impenetrable to our intelligence or to our science, like the phoenix or the chimera? Above all, who will guarantee that reality is conformable with Logic, and that even existing beings do not surreptitiously violate one of its laws? We see that the logical methods employed so light-heartedly by mathematicians quite unexpectedly suggest or point to metaphysical problems of a certain gravity.

VI. LOGIC AND LANGUAGE.

It will not fail to have struck the reader of the preceding paragraphs that logical theories suggest at each step remarks of a grammatical nature. And this is natural, for, to put it briefly, language is but the vulgar and imperfect though the most usual expression of the thought of which Logic seeks to determine the laws. Nevertheless, the relations between Logic and language have been generally neglected by philosophers. If we are to be guided by their scholastic programmes, they are occupied at most with one sole question, *i.e.* the origin of language. This pre-occupation corresponds to an absolutely false and superannuated conception of Philosophy, according to which the object of the latter is "the beginning and the end of things." Such questions (in so far as they are at all soluble) evidently belong to the scientific and historical methods and have nothing really philosophical about them (unless by a confusion of ideas springing from the ambiguity of the word *principium*, "*principle*" is identified with *beginning*). It is equally childish to conceive the relations between Logic and language as do certain nominalists who maintain that Logic is based entirely on the forms of language and who do not even shrink from the extreme and absurd conclusion that there are as many logics as languages.

The true relation between Logic and language has been perfectly indicated by Leibniz: "Languages are the best mirror of the human mind, and an exact analysis of the signification of words would reveal to us, better than any thing else, the operations of the understanding."[1] And he left among his manuscripts numerous attempts at a logical analysis of the forms of language.[2]

[1] *Nouveaux Essais*, iii. vii. end.

[2] See *Opuscules et fragments inédits de Leibniz*, edn. Couturat.

But this branch of research, at once positive and strictly philosophical, has been almost entirely neglected since his time. On the other hand, philologists are generally too preoccupied with the material and physiological part of language (phonetics), and even when they study its intellectual side (in Semantics or the Science of meaning), they are inclined to dwell on the more or less bizarre and illogical particularities (which certainly abound and jump to the eye) rather than to disengage the general features which manifest, in spite of all appearances to the contrary, that there is a latent logic in the formation and evolution of our languages. Philology is too exclusively historical and descriptive, too much in subjection to particular facts; it regards all attempts at appreciation as heresy, and is even averse to all theory.[1] Philologists lack the logical spirit which, essentially critical and *normative*, does not fear to criticise language by confronting it with its aim, *i.e.* the exact and complete expression of thought.

Words are signs for our ideas; they are signs like other signs, but more convenient than others, because they are at once oral and graphic, visible and audible; but still they have to satisfy the conditions which govern all signs. The first of these conditions evidently is that there should be a *univocal* correspondence between the sign and the idea signified; for every idea a single sign and for every sign a single idea. This is the principle of *univocity*, brought to light principally by Ostwald.

This principle is so evident that it seems little more than a hackneyed truism. But its bearing becomes apparent directly we apply it to the critical analysis of our languages. Every notion ought to be expressed in language once and once only (mere economy would counsel this, even if Logic did not). Now the notion of "plural" is repeated five times in the following phrase: "Les bons enfants sont obéissants"; four times by the plural of the article, the adjectives, and the noun, and once again in the plural form of the verb. Similarly the notion of "feminine" is expressed four times in the following phrase: "Une bonne mère est diligente"; once in the idea of mother itself (which ought to be sufficient), and three times more in the article and the adjectives. Again the notion of "person" is

[1] This idolatrous respect for facts goes so far, with certain philologists, that they actually and definitely declare the dialect of any particular place to be *good* and *normal*. It is the apotheosis of *usage* erected into a sovereign and single norm.

always expressed twice in our languages, once by the pronoun (or noun) which is the subject, and a second time by the form of the verb. And here we light on the origin of these pleonasms: it resides in the evolution of our languages which proceeds (speaking roughly) from the synthetic to the analytic. Ancient languages, such as Latin, did not employ the subject-pronoun with the verb: the person was indicated by the verbal form itself (which had already absorbed a pronoun, witness to the primitive Greek endings: *mi*, *si*, *ti*...). As these verbal forms weakened and gradually became confused, it was felt necessary to indicate the person more precisely, and a separate pronoun was added, while, at the same time, the personal forms of the verb were preserved.[1] Similarly, case-endings tended at one period (to a certain degree) to replace prepositions themselves, and came from older agglutinated prepositions. But their meaning gradually became confused and faded, and this is why in the classical epoch the Latin of ordinary speech employed prepositions, even with the cases which did not require them. The idea was expressed twice. Nowadays case-endings have nearly disappeared from the Romance languages, the daughters of Latin, and are replaced (advantageously) by prepositions.[2] This is the final result of a logical evolution.

All this perfectly explains the pleonasms which encumber our languages, but does not justify them from the logical point of view. Moreover, we see that the popular and unconscious logic which presides over the evolution of our languages tends to eliminate progressively double uses and superfluities. Conscious logic, therefore, would only be anticipating natural evolution if it suppressed them from now onwards.

By an inverse phenomenon, but in virtue of the same interior logic, our languages tend to create special words to express certain ideas which lack proper expression. For example, interrogation has, in our languages, no proper expression (such as have negation, doubt, etc.), except the inversion of the subject, which is an inconvenient and insecure proceeding. This is why many languages have forged special words or locutions to give special expression to this idea; for

[1] It is a reduplicative phenomenon analogous to that which has engendered in French *aujourd'hui* (*hui*=*hodie*) and in vulgar French: *au jour d'aujourd'hui*!

[2] Except in certain cases, *e.g.* "Je *lui* donne" and "Je donne *à lui*." The form "*lui*" is a dative which becomes useless with the preposition *à*.

example, the English *do* (they no longer say, "dream I?" but, "do I dream?"), the Danish *mon*, the French *est-ce que*. And in vulgar French a very convenient interrogative particle has made its appearance: *ti*, *e.g.* *je sais-ti?* *j'ai-ti couru?* (taken, by analogy, from the third person, *est-il venu?*).[1]

Thus the immanent logic of our languages ceaselessly tends to apply the principle of *univocity*, or at least of approximating to it. But it is constantly impeded by custom and tradition, *i.e.* by the secular products of evolution which every language bears within it. Our modern languages, even those most highly evolved, carry profound traces of prehistoric (and prelogical) mentality, and they will only disengage themselves from these very slowly and very incompletely. It is only in an artificial language that we can wipe out the past; only there could we apply in all its rigour the principle of univocity, and hope to realise the desiderata of Logic. Few people have an idea to what a degree of simplicity such a language could be reduced, while at the same time it would provide as adequately, and even more than do our traditional languages, all the elements necessary for the exact and precise expression of thought.[2]

It goes without saying that the principle of univocity should be applied not only to grammatical inflexions, but also to the meaning of separate words, especially to particles (prepositions and conjunctions). Few realise the clearness and precision which would characterise a language in which each particle had a perfectly definite meaning, and one only, whereas in all our languages every particle has a crowd of meanings and different uses, determined solely by caprice and "use."

But it is above all in derivation, when it ought to be applied with the most rigour, that the principle of univocity is most constantly violated.[3] In principle nothing is more logical or more convenient than the system of derivation of the Indo-European languages; to roots expressing certain notions prefixes and suffixes are added which express certain constant and well-defined relations, the *Atrides* are the

[1] See Jespersen, *Progress in Language*, p. 93 ff.

[2] For example, it is perfectly logical and very convenient to have the three tenses not only in the indicative, but also in the infinitive and the participle; this gives resources comparable and superior to those of ancient Greek.

[3] See Couturat, *Étude sur la dérivation dans la langue internationale* (Paris: Delagrave, 1910).

descendants of *Atreus*, the *Pelopides* the descendants of *Pelops*, etc. Confronted with a few examples of this kind even the humblest and least logical mind would understand that *id* is the suffix which indicates *descendant of*. To draw this inference there is only required a feeling for analogy, which is the instructive and popular form of logic. But before we can apply it to our prefixes and suffixes it is necessary, first, that they should be invariable in form, and secondly, invariable in sense. Now this is not the case in our languages; sometimes the same affix is employed in different senses, sometimes the same logical relation is expressed by different affixes. *Mange-able*, *pot-able* signify what can be eaten or drunk, but *aim-able*, *admir-able*, *estim-able*, *respect-able* signify what *ought to be* loved, admired, etc.... The names of professionals or artists are derived from a crowd of different suffixes: *art-iste*, *dent-iste*, *pian-iste*, *serrur-ier*, *charpent-ier*, *bott-ier*, *pharmac-ien*, *char-ron*, *forger-on*, etc. And the same suffixes, moreover, seem to indicate quite different relations: that which contains, *encr-ier*, *plum-ier*; the inhabitant of, *Brésil-ien*, *Paris-ien*. There are, it is true, partial series in which the analogy is carried out, but they cross one another and are confused together in such a way that the original regularity is no more apparent. It is evident that if the principle of univocity is to be respected, we must employ the same and only suffix (*-ist*) for the professional, another (*-ny*) for the recipient, another for the inhabitant (*-an*), and so on. A language so constructed would be infinitely clearer, more logical, and more regular than our "natural" languages. It is true it would be "artificial," but neither more nor less so than the nomenclature of chemistry or many other technical terminologies.

The study of derivation (as it exists in Indo-European languages) leads us to establish an essential and fundamental distinction between two classes of words, or rather roots; *nominal* roots which signify beings or objects, and *verbal* roots which signify actions, states or relations. This distinction corresponds roughly to that of *classes* (concepts) and relations.[1] The second give rise directly to verbs: the first engender directly nouns, *i.e.* substantives or adjectives. This explains the close affinity between substantives and adjectives: a word

[1] There are, it is true, nominal roots which express relation: *père*, *chef*, *égal*, *semblable*, etc. But they express it under the form of a *quality* inherent in an object, and hence again applicable to an object.

passes very easily from one class to another: *avare*, *aveugle*, *veuve*, (une) *belle*, (une) *blonde*, etc. On the other hand, the verbal roots form a class essentially distinct: *dormir*, *parler*, *courir*, *aimer*. It is true these words can be transformed into substantives: *sommeil*, *parole*, *course*, *amour*, but such substantives simply express *the fact of* sleeping, speaking, etc., they present the action under the form of an object (of a concept), stripped of the element of assertion which the verb implies. They are, in fact, equivalent to the *infinitive* which some languages substantify directly (*das Rennen*, *das Sprechen*, *le manger*, *le boire*, *le dormir*).

We are thus led to distinguish between *immediate* and *mediate* derivation: the latter is effected by means of affixes (prefixes or suffixes). In immediate derivation, when no affix appears, the root preserves its meaning (in virtue of the principle of univocity); and this is why an adjective becomes a substantive *of the same meaning*; and the verb also engenders immediately a substantive *of the same meaning*, *i.e.* expressing the verbal idea itself (to love, love; to esteem, esteem; to walk, walk). But the name of an object cannot be derived *directly* from a verb, nor a verb from the name of an object; this is evident, for they are heterogeneous notions. This logical consequence is confirmed by the comparative study of our languages; they all possess participles which are derived from the verbal root by means of some suffix. Now what is a participle? It is a noun derived from a verb, and this noun signifies in the active the subject doing the action, and in the passive, the object which suffers it. "Le *mendiant* est l'homme qui mendie; *l'envoyé* est l'homme qu'on envoie." The same relation between noun and verb is indicated by other suffixes (*chant-eur expédit-eur*, *recev-eur*). But a suffix is always necessary to derive these names from verbs, and they must not be confused with *verbal substantives*, which signify action (*chant*, *envoi*, etc.).

Inversely, we cannot immediately derive a verb from the name of an object, for the same logical reason: and it is here that our languages sin most frequently and most seriously against logic. *Patronner = être patron*, *aveugler = rendre aveugle*, *couronner = orner d'une couronne; saler = ajouter du sel; plumer = enlever les plumes; fleurir =* (1) *produire des fleurs*, (2) *garnir de fleurs*, *etc.* In a word, these "immediate" derivations

express a crowd of diverse and even contrary relations.[1] This is contrary to the principle of univocity; in order to satisfy this principle each derivation having a special meaning must be effected by a special affix, or, more generally, to each element of the idea there must be a corresponding word-element. To be a patron=*patron-es-ar*, to render blind is *blind-igar*, to provide with a crown or with salt is *kron-izar*, *sal-izar*, to produce flowers or fruit is *flor-ifar*, *frukt-ifar* (to fructify), to deprive of feathers means "to render featherless," and=*sen-plum-igar*, etc. Thus we obtain a perfect logical derivation, universally clear and hence international (in spite of the example to the contrary of our languages which swarm with idioms of derivation).

Such a derivation ought to verify, and does as a matter of fact verify, the *principle of reversibility*, which is a corollary of the principle of univocity. This principle may be stated as follows: To every derivation of meaning there ought to be a corresponding derivation *of form*, *i.e.* the addition or suppression of an element of the word, for if we can pass from one word to another in virtue of a certain rule we ought to be able to pass from the second to the first in virtue of the inverse rule. For example (to quote the most important application of this principle) if the substantive immediately derived from a verb signifies action or state, inversely the verb derived immediately from a substantive can mean nothing but "to do that particular action" or "to be in that state." Thus from *paco*=peace, we can derive the verb *pacar*=to be at peace; from *muziko*=music, we can get *muzikar*=to make music; for what is the fact of being at peace, but peace; and the fact of making music, if not music? But from *krono*=crown, we cannot derive *kronar*=to crown, for from this verb we derive, inversely, *krono*=coronation, and we should then have two meanings for the same word. Similarly, from *domo*=home, we cannot derive the verb *domar*=to stay at home, for then *domo* would mean the fact of staying at home. From *bela* (beautiful) we cannot derive *belar*=to be beautiful, for then *belo* would be beauty and not a beautiful being; and so on. We see that the principle of reversibility is a sure criterion (and practically a very convenient one) in verifying the logical value of a derivation.

[1] We may add that these relations may differ between one language and another; *e.g.* Fr. *documenter*=provide with documents, while the German *dokumentieren*=to prove by documents.

But we must not turn aside here to discuss logical derivation in detail. There is one form of derivation, however, to which I should like to call the attention of philosophers: it is that which binds an adjective of quality to the noun of the same quality. It seems to certain minds (led astray as they are by the example of their national language) illogical that since *bela* means beautiful, beauty should be rendered by *belo* (*i.e. bela* substantified). But this is a serious error. *Bela*, like all qualifying adjectives, is a class-concept which applies to all beautiful individuals, but the abstract quality of "beautiful" is not a beautiful object, it is the fact of being beautiful; and just as *to be beautiful* must be expressed by *bel-esar*, and not *belar*, so the fact of being beautiful must be rendered *bel-eso*. Moreover, this is more conformable with the logic immanent in our languages, which all have a special suffix in order to derive abstract quality, *e.g.* G. *Schön-heit*, E. *ill-ness*, F. *rich-esse*, I. *bell-ezza*, etc. No doubt primitive languages did not have the verb *to be*, and expressed *I am beautiful* by *Me beautiful*. But the invention of the verb *to be* is one of the conquests of the logical spirit, and the more abstract this notion is the later its acquisition. To suppress or neglect this essential element of civilized language and the suffix which is its equivalent would be to mutilate logical thought.

It is, moreover, this element which serves to transform a noun into an attributive verb, into a "predication." Doubtless we must not take the traditional analysis literally. I sing = I am singing, according to which all verbs of action or state are transformed into attributive verbs. It is no less true that there is an equivalence (if not identity of meaning) between attributive verbs and the others.

The element *es* (root of the verb *être*) is the inverse or converse of the suffix of participles (when this is *-ant*). For indeed, as we have seen, the latter serves inversely to transform a verb into a noun (qualifying adjective or substantive: *parolanta* = *parlant*, *parolanto* = *orateur*, *i.e.* he who speaks). According to the analysis quoted above we have the following equivalence, *me kant-as* = *me es-as kant-anta*. If we abstract the elements common to the two members, we find: *es* + *anta* = 0. This can be confirmed in another way: *bel-esar* = *être beau* (to be beautiful); *bel-es-anta* = *étant beau* (being beautiful) = *beau* (beautiful) = *bela*. Thus *es* + *ant* = 0, and these two elements cancel one another.

Moreover, the relative pronoun (*qua*) plays exactly the same rôle as the suffix of the participle, as is proved by the synonym: *aimant*=*qui aime* (lover=he who loves), chantant=*qui chante*. Consequently, it too is the reciprocal (*réciproque*) of the verb *to be*: *qua esas bela* = *bela* simply; *qua* and *esas* cancelling one another. As we have said, the effect of the relative pronoun is to transform a proposition into a concept, into an epithet: who speaks is the orator; who sings, the singer. It results from this that all adjectives can be immediately substantified; who is eternal is the Eternal ("who is" = o). Those classical but vague expressions, *the beautiful*, *the good*, *the true*, really mean what is beautiful, good, true, *i.e.* objects. We may say, after Plato, that these expressions signify the *essence* of the beautiful, the good, the true, in other words, the abstract qualities, beauty, goodness, truth (it matters little whether Plato did or did not conceive them as existing apart from beautiful, good, etc., objects; that does not affect their nature). Plato himself insisted on the point, paradoxical but evident after what we have said, that the *beautiful in itself*, the *essence of the beautiful*, cannot be qualified as beautiful; it is not itself beautiful, but is that which causes to be beautiful, and is thus the cause or "form" of the beautiful.

We see that these essays at grammatical and logical analysis, disdained by certain philosophers, may sometimes, as Leibniz foresaw, penetrate to the intimate foundations of language and lead to the analysis of forms of thought. It is therefore no unworthy task for a philosopher to collaborate in the institution of a language which joins to the theoretical advantage of being logical and clear the immense practical advantage of being infinitely easier than any national language, of being "*the easiest for the greatest number of men*"; while it would also furnish us with an instrument of international communication more convenient and more perfect, in certain respects, than our languages. As that illustrious philologist, H. Schuchardt, has said, an international language is a scientific as well as a practical desideratum. Is not the language of the sciences to a great extent artificial? and is not every science obliged to elaborate its language in proportion to its development? Such a language, then, answers to the highest needs of the mind as well as to those of ordinary life; it tends to realize the ideal of human language at which our languages are only confused and complicated attempts,

according to the profound saying, "Was die Sprache gewollt, haben die Sprachen zerstört."[1]

Can we doubt that existing languages realize only very imperfectly the ideal of language? Language, too long regarded by certain savants with a superstitious and almost mystic respect, is, after all, nothing but one instrument (amongst others) of thought; thought can and ought to fashion and modify it according to its own needs and convenience, and if linguistics teaches us how languages have been, as a *matter of fact*, formed and evolved, Logic teaches us what language ought to be if it is to serve as an adequate expression of thought. Doubtless observation and an exact analysis of the forms of language can teach us much as to the mechanism of thought. But the human mind has the right to perfect this instrument, like all others, and to make it fit for the end it has to serve. Here Logic, like all the other sciences, can find a practical application, and while contributing to the elaboration of a language truly international and rational, may also further the amelioration of human life and the progress of civilization.

[1] See *La Langue internationale et la science*, by Couturat, Jespersen, Lorenz, Ostwald and Pfander (Paris: Delagrave, 1909). *Weltsprache und Wissenschaft* (Jena: G. Fischer, 1909). *International Language and Science* (London: Constable, 1910). *Varldsspràk och Vetenskap* (Stockholm: Bagge, 1910).

THE TASK OF LOGIC

BY

BENEDETTO CROCE.

UNDER the present condition of things, a writer who has the interests of Philosophy at heart is placed in a position of some difficulty when invited to express his views on logical problems. In this case, perhaps my wisest course will be to start by discussing the different meanings in which the term "Logic" has been understood, and explaining in what sense I myself propose to take it. For I shall then at any rate avoid the dangers which attend a *naïve* exposition of personal views, and shall perhaps save myself the mortification of meeting with an unfavourable reception. Nor shall I be open to the reproach that nothing I have found to say conduces in the very least to the furtherance of the Sciences (by which are meant the Natural Sciences and Mathematics). Moreover, I shall, I hope, have guarded against the risk of disappointing and wounding certain deserving authors—authors who have been at endless pains to provide their fellows with ready-made and sure logical instruments, and have even aspired to endowing them with an entirely new language, at once simple, exact and *universal*. I refer, of course, to the *Logisticians*. According to the view taken by these gentlemen, the task of Logic consists in drawing up an inventory of rules and formulae which are to illuminate the discussion and investigation of truth. They may therefore be regarded as the lineal descendants and representatives of the scholastic *logica utens*. Such a Logic is not altogether without its uses; but we must not forget the circumstances which attended its blossoming time, or, to speak more correctly, the time at which it spread out its thorns towards the sun. Philosophical controversy had then become so external and empty, had

descended to such pedantic and tiresome quibbling, that soon afterwards an insurrection arose among the spirits it had held captive. The result of this insurrection was twofold: as regards subject-matter, it effected a return to direct observation, to experiment, to original sources and to analysis; while, with regard to form, the free-and-easy speech of the "man of the world" was preferred before the gnarled and rugged expressions of the monastic dialectician. Nevertheless, we honestly hope that the modern Logistic will at least succeed in preserving the advantages, small and unimportant though these may have been, that "formal" inference brought about in its time. So far this hope has not been realized. Hence the Logisticians of to-day are in sorry case; they are like pedlars who carry round specious and glittering wares, crying them up to the public and advertising their utility for all sorts of purposes, but who fail to attract customers. Such misfortune embitters them. And this is no wonder, for these benefactors see their warmest wish, namely, to benefit mankind, set at nought thereby. Unfortunately we do not feel able to set a good example by ourselves buying of their wares. It has long been our intention and habit to express ourselves decently and in a comprehensible manner. The austerity of the new formulae frightens us. We will leave it to a younger and stronger generation to appropriate them.

One point at any rate is certain. Whatever practical value Logistic may embody, however rich a future may await it, as *logica utens* it can never be *logica docens*; as *practice* it cannot be *theory*; as a *complex of rules and formulae* it is not a *science*. According to our view, on the other hand, Logic is essentially a *doctrine*, a *theory*, a *science*. It is no part of its business to assist thought,—to further the progress of Natural Science, Mathematics or any of the special sciences, to facilitate research or to simplify the art of disputation. It is a theory entirely devoted to the task of inquiring into the nature of thought, as exemplified in science as a whole and in the particular sciences. No one acquainted with the promoters and editors of this Encyclopaedia will doubt for a moment that, when they invited me to contribute an article on Logic, they meant Logic in this *theoretical sense*. I need not therefore pause to elaborate the answer to the question I raised in my opening paragraph. If I have touched upon it in passing, it was only in order to

delimit the object of my inquiry and make it quite clear. Nor must the Logisticians expect to find anything in our inquiry which runs counter to their own views. For there is no room for opposition where the objects pursued are entirely different. And such entirely disparate objects are reflexion and tactics, concept and instrument, even when the latter is the good sharp sword of the Logistician. We should also like to greet in passing those normative and formal logicians who, in prescribing norms, practically prepared the way for the more radical demands of the Logisticians themselves.

But we cannot rest satisfied with asserting the right of Logic to be recognized as a science; we must make the further demand that Philosophy alone—and not the empirical sciences—be admitted as science in the strict sense of the word. Since, however, this is not the place in which to discuss the wider and narrower senses in which the word "science" may be taken, we must content ourselves here with asserting that Logic, as we conceive it, is a *philosophical science*; and in making this statement we have at once excluded another side of the traditional scholastic Logic, indeed of Logistic itself. I refer to those expositions which profess to be a theory or science of thinking, while what they actually offer is nothing more than a description of the different forms of thought, such, for example, as the concept, the judgment, abridged and compound syllogisms, modes, figures, etc. It may not be out of place to repeat here what we said before—*à propos* of the rules and formulae of the Logistician; as we were far from wishing to deny to Logistic the right to exist, so too we have no wish to see *Descriptive Logic* (which has been incorrectly termed "Formal Logic") abolished. But we should like to point out the limitations common to Descriptive Logic and every other descriptive science, whatever be the aspect of spirit or reality with which they are concerned. Such sciences can, at best, but furnish us with convenient summaries, which are as useful for the memory as they are insignificant and uninforming for the understanding. Indeed, these summaries may become positively injurious if, as often happens, they are mistaken for true science and their purely nominal concepts are believed to be real concepts.

Without entering on any complicated epistemological discussions, we will pass on to consider the fundamental trichotomy

into concept, judgment and conclusion. This division involves the assumption that three different moments can be distinguished within what is really a single and unanalysable act of thought. As a matter of fact, no one will ever succeed in thinking a concept, a real concept, which is not at the same time a judgment, that is to say, an assertion of its own essence; nor can any one find a concept or a judgment which is not at the same time a conclusion, being connected in a system with other conceptions and judgments. The apparent justification for the analysis into three of a single unanalysable act of thought lies in the fact that, in place of living thought as it wells up from its source have been substituted names, sentences, and all the other fragments into which speech is broken up by Grammar. But this mosaic is not the act of speaking: speech is no aggregate of stones held together externally by cement, it is rather a flowing river, in which one wave overtakes another. The difficulty of drawing any sharp line between *Formal Logic* and *Logical Grammar* has been frequently remarked upon. And this need not surprise us, for, being descriptive and empirical sciences, they both proceed to do violence after the manner of their kind, schematizing acts of thought, and arranging them according to accidental and superficial resemblances. And no more than in the case of Grammar can Formal Logic mark itself off by any sharp line of division from the psychology of cognition, or *Psychological Logic*, unless it be that such a principle of division is contained in the normative element present in the latter. But this normative element is not of the essence of the psychology of cognition; it may very well be absent. So that if we show ourselves tolerant towards formal, verbal or empirical Logic—and how, as we have already remarked, can we deny the right of existence to any product of the human spirit?—it is only under certain conditions. It must be borne in mind not only that such a Logic has nothing in common with the true science of Logic, but also that it can never be co-ordinated with philosophical Logic at all. It is not a further and legitimate method of treating Logic, a kind of propaedeutic or completion of it in its philosophical form, but is something altogether different. Philosophical Logic must pursue its own way, regardless of this other Logic, except where the latter steps outside its own province to breed errors and nurture prejudices; then, indeed, it must be combated. It is the

custom nowadays to emphasize the opposition between empirical and philosophical Logic—more especially when the former is clothed in the garment of Psychological Logic—by calling empirical Logic the science of *reality* and of *facts*, while to philosophical Logic has been assigned the title: Science of *values*. But we suspect another reason for this recourse to the term "value" on the part not only of Logic, but also of the other philosophical sciences. We believe they have been forced to this because Empiricism has outwitted them, and has unlawfully possessed herself of the names "reality" and "fact" in order to bestow them upon her classificatory concepts and abstractions, which, as a matter of fact, are *un*realities. It is as though an honest man were obliged to change his name because it had fallen into disrepute through the criminal behaviour of a member of his family. The "value" which is the object of philosophical Logic is logical reality and fact itself: it is value and norm intrinsically, as existing and working. It is now high time for the real Logic to claim her own again; nor does she owe any consideration to the Empiricists and Positivists. Concessions in this direction, even though purely verbal, would be a serious mistake. Let Logic once more assert her claim to firm *facts* and cease to content herself—whether out of pride or modesty—with anything so unsubstantial as *values*.

But we must now turn to the consideration of another difficulty. While Logic has been defined as a special philosophical science on the one hand, it has been maintained on the other that Philosophy is an undifferentiated unity, within which the universal cannot be discriminated from the particular, the edifice from the foundation, the upper from the lower storeys. And yet, of course, such distinctions are as customary to-day as they were in the past; we still speak, for instance, of a general and a special philosophy, of an introduction and systematic part, of a Logic or Epistemology which is to supply the criterion and method of investigation, and of a Metaphysic which is to crown and round off the investigation itself. But these divisions and distinctions must not be taken strictly. Our only justification for their employment is based on literary and educational considerations, and applies within these spheres alone. Logic as a special science is unthinkable and incomprehensible, because it can only be thought and understood through the whole of

which it is an inseparable part. So, too, thinking apart from being is incomprehensible; as incomprehensible as is knowing apart from willing, reason (the Logos) apart from imagination, the concrete apart from the abstract, the individual apart from the universal. But what is this but to say that Logic as philosophical science is Philosophy itself? And if this is so, why do we speak of it as *one of* the philosophical sciences? There is only one way out of this difficulty: we must admit that when we come really to think out and expound Logic, or any other special philosophical science—as, for instance, Ethics and Æsthetics—each in turn proves to be the whole of Philosophy. And indeed it is only for educational purposes that Logic should be treated apart from the other special philosophical sciences, and from Philosophy as a whole. The books in which it is thus preferred before other disciplines should be recognizable as text-books from their titles. It is only by bearing this constantly in mind that we can hope to preserve Logic itself from confusions. Philosophical *specialism* was an error of the last decade; it went hand in hand with the discredit into which Philosophy fell at that time. It is specialism that it is responsible for the slackness and indolence which then invaded Philosophy. But to-day we are forced to recognize the unanalysable unity of Philosophy; we admit the necessity of thinking the particular through the universal, and of holding fast to the inner bond by which the distinct moments are united. If we are tempted to doubt the necessity of this interconnexion, we have only to remind ourselves of the mass of absurdities and confusions which the specialists introduced into the sphere of Logic by their neglect and ignorance of other philosophical problems; such problems, for instance, as are involved in speech, in Art, in historical investigation, in the economic will, etc. They hardly paid any attention at all to these problems; and when they did so, it was only to take over ready-made solutions of them from other specialists; these they proceeded to cram into their books without ever troubling to investigate their relation to Logic itself. It is not difficult to trace the results of such a procedure: an aggregating, a setting side by side, a cementing together of fragments of philosophy which, apart from their inner unity, had no further life. We take up the cudgels here against the *separatists*, as we did just now against the practicians and empiricists.

One result effected by this separatism was the withdrawal of so-called Elementary Logic from Applied Logic or Methodology, by which was meant the Logic which takes as its object the constitutive forms of knowledge and occupies itself with the knowledge-value of the special sciences. Indeed, this process of tearing apart was carried so far that the problem of the *classification of the sciences* came to be regarded as an independent problem, and was treated as such, apart from all reference to a philosophical system or to any sort of logical basis.

But it is not enough to classify the forms of knowledge; we must also understand their significance. And as soon as this significance was grasped, and the traditional empirical and pedagogic groupings were rectified, it became apparent that these forms were nothing else than the elementary forms of apprehension developed in the course of history to the highest degree of which they were capable. The recognition of this fact was of no small advantage; for with the destruction of the artificial distinctions which divided them the elementary forms lose their abstractness, gain sap and force, and immediately prove fruitful, while the Forms of Knowledge lay aside their clumsy obscurity and reveal to us their real nature. Logic itself, which had previously been regarded as the driest of all the sciences, the furthest removed from the problems of daily life, was now seen to be intimately related with these problems, influencing them and being influenced by them in turn. And, best of all, we have got rid of the absurd idea of an "applicable" and "applied" Logic; for it has become evident that the activities of spirit are not applied but develop, and that in this development they preserve intact both their origin and their essence. We learn, moreover, that there can be no spiritual products issuing from heterogeneous elements or from the working up of external and given materials.

Now that we are convinced of the identity of the elementary forms of cognition with the elaborated forms employed by Science, we can start our inquiry as to their epistemological basis from one side or the other, as may happen to suit us; because, as we have said, they are distinguished only in an external way. It can make no difference to the thing itself whether we first consider the representation (*Vorstellungsbegriff*) or classificatory concept, and then pass on to the natural sciences, or whether, inversely, we begin with the natural

sciences, and proceed from them to the classificatory concept. For to determine a generic concept is to establish a piece of knowledge within the natural sciences, and *vice versa.* Both ways lead to the same goal; indeed, they are really one and the same. But it will conduce to clearness here, where our object is to give a survey of Logic as we conceive it, if we start from the developed forms of knowledge. Of these forms we will take four, the difference between which is generally recognized: the *Art of Poetry* (representing Art in general), *Philosophy*, *Natural Science* and *Mathematics.*

The question, however, has been raised as to whether these four forms of knowledge cannot be further reduced, and this has indeed been attempted in the case of some of them, though in my opinion unsuccessfully. It would carry us too far out of our way here to go into this question in detail, and I have done it elsewhere;[1] I shall therefore content myself with stating my case, without proving it. Thus, for example, the Art of Poetry was classed under the sciences by Bacon and Hobbes; then it was taken out of this classification altogether, but was afterwards re-introduced in conjunction with Religion as a moment of absolute spirit or of Philosophy; then once more it was taken out and ascribed to feeling, under the head of "play." It was further confused with a kind of popular Philosophy, and supposed to disseminate concepts and type-ideas. But Poetry gives us what no other form of cognition can give, namely, the *naïve* acceptance of Reality, without either the analysis or the synthesis of reflexion. In Poetry we meet Reality in that virginal form to which we must turn back again and again from its more complicated aspects, in order to refresh our memory and to grasp new features. It is to this great *rôle* which she is called upon to play, that Poetry owes the place of honour which she occupies in the life of humanity, both in education and in adult life. This would be inexplicable if her task were restricted to duplicating Science or Philosophy, or if she were indeed a kind of game, or, still worse, if she ranked among the confused and sensuous aberrations of feeling.

Philosophy has been subjected to similar experiences. Her real character has been impugned again and again. At one time the Positivists attempted to show that she was only a part

[1] See Benedetto Croce, *Logica come Scienza del concetto puro.* 2nd edition. Laterza et Figli, Bari. 1909.

of Natural Science and Mathematics; at another, her task was restricted to the derogatory office of putting together the results attained by the special sciences, or she was assigned a place within the Art of Poetry as her only refuge; for men saw in Philosophy merely a complex of beautiful fancies, which was only to be tolerated as such and could claim no further recognition. Hence we must be prepared in advance for what is sure to happen at the Philosophical Congress, shortly to be held in Bologna:[1] certain scientists and mathematicians will not be able to resist the temptation of serving up their beautifully formulated views on Philosophy; we shall have the step-sisterly sciences raising their voices against the modern Cinderella, and ambiguous compliments will be paid to the new *folle du logis.* But we have an answer to all these attacks. We need only point to the history of Philosophy, to its continuous development through the centuries. This furnishes us with a brilliant exhibition of what Philosophy was, is and will be: it teaches us that it is neither the Art of Poetry, nor the Science of Nature, nor even Mathematics, but the search after the One and the All—*amor Dei intellectualis.* Lastly, we may remind our readers that the attempted reduction of Philosophy to Natural Science or to a mathematical Science of Nature has foundered, and is to-day entirely abandoned; for Mathematics found itself, as was inevitable, brought up against its prescribed limitations (the indispensable perceptual element), which constitute not only its limitations, but also the range of the possibility of its application.

Corresponding to the four forms of knowledge which we have enumerated are four elementary forms: *perception*, the *philosophical concept* or the *Idea*, the *scientific* or *classificatory concept* and the *mathematical* or *abstract concept*; or, according to the names of the corresponding activities, creative imagination or intuition, thought, classification and abstraction. They all denote one and the same thing, looked at from different standpoints and therefore differently designated. And we may here join with Schleiermacher in issuing a warning against any sort of epistemological *aristocracy.* We should rather strive to establish a kind of *democracy*, pointing out that

[1] This treatise was presented before the Philosophical Congress held at Bologna in April, 1911, to which distinguished physicists, chemists and astronomers were invited.

Poetry, Philosophy, Natural Science and Mathematics are no wonderful and special businesses, suitable for high days and holidays only, but, on the contrary, that they constitute the modest daily round of every thinking being. For no one can live without at every moment poetizing, thinking, classifying and abstracting. And this is really the reason why the highest manifestations of these faculties—the rare works of originality in the spheres of Art, Philosophy, Mathematics and Natural Science—evoke in us such great admiration. For these magnificent products come as it were to meet our minds; they blend with them and dissolve into them, and by so doing strengthen and elevate them.

If now we make it our business to discover which of the four forms of knowledge constitutes the proper subject-matter of Logic, we must begin by excluding the first form, Poetry, perception, intuition, imagination; for the logical subject-matter implies generality, and the poetic form does not pass beyond the particularity of the individual. Hence the art of Poetry finds no place in a discussion of logical problems. As a special sphere of knowledge it does not belong here, but to Æsthetics, which, as is well known, was raised by Vico to a science under the name of *Logica Poetica*, while Breitinger called it the "Logic of the imagination," and Baumgarten *Ars analogi rationis* and *gnoseologia inferior*. Of course thought void of intuition or imagination, unexpressed and unspoken, were itself unthinkable; and we have drawn attention elsewhere to the deplorable effects of the entire neglect of aesthetic problems on the part of logicians. Of course, Æsthetic conditions Logic; but for that very reason it cannot *qua* Æsthetic constitute the specific logical moment.

It will be easy enough (too easy, we might say, for the contrary were rather to be wished) to usher Poetry out of the domain of Logic without encountering opposition on the part of modern logicians, bitten, as most of them are, with intellectualism. We shall have a much more difficult task to convince them that neither do classificatory and abstract concepts, the concepts of Natural Science and Mathematics, fall within the proper sphere of Logic. Nevertheless, a searching examination of those concepts and sciences has shown clearly and distinctly that they are inadequate to Reality. If scientific concepts, by means of simplification, take up certain sides of

Reality, and make out of them symbols and fictions, mathematical concepts go entirely beyond Reality and create a whole world of empty figments; both alike embody not a logical but a practical moment, which has rightly been called "economic." This epistemological inquiry into Natural Science and Mathematics (which latter may be regarded as the modern and more developed form of Nominalism) has shown us that true knowledge must lie beyond these sciences. If we are not to renounce it altogether as beyond the reach of man, or to fall into some other kind of Agnosticism, we must seek it under some other form. What sometimes actually happens is that a Realism, basing itself on intuition or pure experience, springs up as a counterpart and make-weight to this Nominalism. But we ourselves, since we have already indicated, on logical grounds, the inadequacy of Poetry, must expect to find this knowledge only under the second in our list of general forms, which indeed is the only one that really gives us the universal: I mean the philosophical concept or Idea, or, in other words, Philosophy itself, which is the fundamental faculty of the true universals. And we do this with all the greater certainty, because the fictions of Natural Science and of Mathematics necessarily point beyond themselves to the concept of a concept which is not in its turn a simulation of something else. As we have had occasion to remark elsewhere, as false money presupposes true, so the abstract presupposes the concrete, and the arbitrary the necessary. But Logic as the science of knowledge is saved by the very nature of its object from being a science of *nomina* and fictions. It is the science of true science or of the philosophical concept; in other words, it is the philosophy of Philosophy.

We need, however, have no fear that Logic will not know how to treat the empirical and abstract concepts and the sciences which correspond to them. She has, on several grounds, already shown herself capable of excluding Poetry from her own domain, while assigning to her her rightful place, and indicating clearly the relation between image and concept, the individual and the universal. So now she will regard it as her duty to give their due to these sciences and their concepts, and to indicate the relations which exist between them and Philosophy. These relations may be compared to those obtaining between a librarian and readers. Natural Science and

Mathematics reach down the books from the shelves, and display great skill in providing clues by which any particular book can be found in the quickest and surest manner possible. But they themselves neither write nor read these books; and were they to arrogate to themselves the right of conducting the march of human knowledge they would proclaim themselves the enemies of all thinking people. For they would demand of the latter that they should give up thinking according to the principles of truth, and be guided instead by the arrangement of book-shelves and catalogues.

It may be objected that in limiting the forms and pseudo-forms of knowledge to four we have forgotten that of Religion. But this is not the case. It ought to be clear from the foregoing that we have assumed throughout that the identity of Religion and Philosophy has been both exhibited and recognized: not indeed by resolving Philosophy into Religion, but Religion into Philosophy, so that Philosophy participates in the value of true and complete Religion. We are, however, open to the reproach of having overlooked a fifth form of knowledge, which plays a prominent part in scientific inquiry, held by theoretical minds to be in the highest degree essential. We refer to the individual judgment or judgment of fact, and to History, the scientific form which corresponds to it. Historical knowledge obviously differs in kind from mathematical or scientific knowledge; for, while Natural Science schematizes and classifies, History individualizes and narrates. If the former seeks the typical in the manifold, History, on the other hand, separates the manifold from the typical. But though History, in virtue of this individualizing and narrating, approximates so closely to Poetry, she must not be identified with the art of Poetry, pure and simple. For History bears a realistic stamp, and employs a concept of reality of which Poetry is ignorant. History asserts that this and not that took place, while Poetry knows nothing about real and unreal, the actual and the possible. She is beyond such categories, and is entirely absorbed in the ideal world of creative imagination.

We are thus confronted by the following alternative: History must either be recognized as a form of spirit existing in and for itself, or it must be identified with Philosophy. In the first case we should have two concrete and absolutely true forms of knowledge—Philosophy and History—and two elementary

forms corresponding to them, namely, the concept and the historical or individual judgment, or, since we have seen that every concept is also a judgment, a general and an individual judgment. But such a conclusion would be satisfactory to no one, for it is inconceivable that truth should split itself up into two distinct orders. Before we accept a fact so contrary to reason, let us consider the remaining alternative, according to which History and Philosophy, the individual and the general judgment (or definition) are identical. Perhaps we shall find that this can be maintained. One thing at any rate seems certain from the beginning, and that is, that History is intimately connected with Philosophy; indeed its relation to it is one of necessary dependence, for it is impossible to narrate History, to pass a single individual judgment—even of the most trifling character—without employing concepts, that is to say, Philosophy. If a man is to narrate History, he must begin by understanding it, and he can only do this by bringing into consciousness the ideas which lie concealed within it. Historical objectivity has meaning only in so far as it is contrasted with that passionate subjectivity which is capable of distorting historical truth, and appeals to that higher form of subjectivity which must only be ascribed to the concept or philosophical Idea. But this dependence of History on Philosophy, in virtue of which every advance in Philosophy necessarily implies an advance in the understanding of History, does not amount to a proof of their identity. According to this way of looking at them, Philosophy is still absorbed in excogitating an extra-historical concept; and yet she willingly lends to History, which could not live without it, the light that she has gained by means of this excogitation. How are we to understand this? We can only grasp their complete identity by placing ourselves on the soil of the great idealistic Philosophy of the first half of the nineteenth century; we must set aside the old notion that Philosophy is a fixed and immovable rendering of the unchanging, and realize that Philosophy itself is History. In truth, Philosophy is nothing else than the eternal solving of problems which are always different, and yet which always spring out of the womb of actual History (History *a parte objecti*). Hence Philosophy is at the same time History (*a parte subjecti*). In solving the problems which historical conditions present to her, Philosophy illumines these conditions

themselves; she characterizes them as they are, that is to say, she describes and narrates. Hence every philosophical system is informed by a fresh way of looking at History, the validity and truth of which goes hand in hand with the truth contained in the system; while, on the other hand, every narration of History is impregnated and animated by Philosophy. The further questions as to whether with each new Philosophy the new conception of History must be at once brought forward and separately discussed (as is done by some philosophers), or left till later as a special task, and whether the new way of writing History is aware of the Philosophy immanent within her, these questions, I say, do not in the least endanger the identity which has been laid bare. It behoves us here rather to explain how it is that, in spite of their identity, Philosophy and History may appear so different. This is owing to the literary form of their exposition, in which emphasis is laid on different constitutive elements of the logical form common to both (unity of concept and individual judgment; synthesis *a priori*). Thus, Philosophy emphasizes the universal (the predicate, the category), while History, on the contrary, emphasizes the individual (the subject); and it is this difference in literary emphasis which makes it appear as though Philosophy occupied itself solely with the predicate and History with the subject. But appearance is only appearance, and it remains true that History is Philosophy, and Philosophy History. By recognizing this identity, Philosophy mitigates its own abstractness and History is delivered from the tyranny of its mere material. Thus we see that there is a complete justification, on logical grounds, for the perceptual moment in History, although it has more often led to the assertion of similarity between History and Poetry.

This sketch of the fundamental forms of knowledge contains *in nuce* a considerable part of the Philosophy of spirit, and has already indicated the way in which the different theoretical and theoretico-practical forms arise and succeed one another. Thus we have seen the transition from intuition (Poetry and Art in general) to the concept or individual judgment, by means of which the knowledge of reality is reached and fully developed as Philosophical History. From this point cognition proceeds to the schematization of the knowledge attained by means of classification and the laws of scientific procedure (Natural Science in general, whether of the natural or the spiritual

world), and, finally, to the still further transformation and simplification of these schemata by the help of counting and measuring (the mathematical sciences). But we cannot attempt, in a brief outline of Logic, to develop in detail the main features here indicated, or to justify everything that has been said in passing. All this must form the subject-matter of a more extensive treatment of Logic.[1] There the special problems of Logic must be dealt with, as, for instance, the nature, the marks and the forms of the concept; the doctrines of the Realists and Empiricists, and their eventual reconciliation by means of the admitted duality of the philosophical and scientifico-mathematical concepts, definition, syllogism, perception, the existential predicate, counting and measuring, the principles of Logic; and, finally, the greatest of all questions—that of opposition or dialectic, and the relation between distinct and contrary concepts. And it is only in a work of considerable magnitude that we can hope to find place for a discussion of the problems connected with the so-called Methodology of Philosophy, History, Natural Science and Mathematics: such problems, for instance, as the nature of system and criticism, the question as to the employment of empirical concepts in History or historical factors, the value of Natural Science and of a mathematical science of Nature, etc. These and similar problems which have hitherto had a precarious existence, for the most part in books on Methodology or in the controversies of individual scholars, must now be brought together by Logic in a new focus. The old and useless ballast of Empirical and Formal Logic, which still encumbers the text-books, must be thrown overboard.

But any exposition of a gnoseologically conceived Logic that did not discuss the philosophical theory of error would be guilty of a serious omission. This theory must take the place of those sections, so rich in quibbles, on sophisms and their refutations which the old logic books always contained. Now, error is two-sided. On the one side it is real error, by which I mean those arbitrary conjunctions of words which only appear to state something, but which in reality affirm nothing, for they express no thinkable content; on the other, it is tentative or hypothetical knowledge, and an approximation towards the truth —a partial truth which serves as a stepping-stone to a wider

[1] See Benedetto Croce, *Logica come Scienza del concetto puro.* 2nd edition. Laterza et Figli, Bari. 1909.

truth. Hence the theory of error may be regarded from one side as a *pathology of thought*, and from the other as a *phenomenology of truth.* But, in whichever sense it is taken, it must supply us with a deduction of all the necessary forms of erroneous or incomplete thinking. Such forms arise where the philosophical moment is confused with the other moments of theoretical and practical Spirit; and we get, for example, confusion between concept and imagination, concept and practical life, concept and pseudo-concept, etc. Raised to higher powers, these errors appear under names well known in the history of Philosophy and in philosophical controversy. They are called Scepticism, Mysticism, Dualism, Æstheticism, Empiricism, Mathematicism, Philosophism, etc. As necessary forms, these errors are indispensable and immortal. They perish in every act of thought, and in every act of thought are born anew. For truth has its being only in the struggle against error; hence error supplies it with the nourishment and conditions without which thinking would not be actual thinking, or, in other words, would have no existence. For this reason, because errors constitute these eternal and timeless conditions, we cannot straightway accept Hegel's conception of the history of Philosophy. Hegel was inclined to regard historical periods as embodiments of *ideal forms of error*, and thus to confound the history of Philosophy with the phenomenology of truth.

The doctrine of categories, on the contrary, does not appear to be an integral part of Logic; it was modelled on empirical Logic, and consequently remains connected with it, although the bond between them is a merely external one. Indeed, if we are to understand by the *Categories* the logical forms of thinking Reality, Logic only knows one: the Idea or Concept. In this concept it exhausts its being; hence all those other categories which are usually adduced must be regarded either as mere *nomina* or as not true logical categories. But if, on the contrary, we understand by the categories the elementary and archetypal forms of Spirit and of Reality, their deduction and dialectical genesis, what is this but to say that it takes the whole of Philosophy to furnish a doctrine of categories? Indeed such a doctrine is commensurate with Philosophy itself, for Philosophy, in its essence, is the science of the necessary determinations of Reality; in other words, of the eternal categories. No doubt, as we have already explained, Logic itself provides a doctrine

of categories, because it concentrates within itself, in a more or less developed form, the whole of Philosophy, and coincides with it. In this second sense Logic has rightly assumed, on occasion, the name of Metaphysic. But such a Metaphysic cannot tolerate the existence at its side of any other Philosophy either of Nature or of Spirit; for in so far as such would-be sciences had any real content it would already be comprised in the Metaphysic itself. In order to understand and estimate the value of Hegel's reasons for preserving and systematizing the traditional scholastic division into *philosophia rationalis* (Logic and Metaphysic) and *philosophia realis* (the philosophy of Nature and of Spirit), the whole of the Hegelian philosophy must be more thoroughly examined—a necessity which is becoming daily more urgent. This cannot, however, be undertaken here. But I can find no better conclusion for this brief essay than the expression of the hope that the as yet untouched treasures of the Hegelian Philosophy may speedily be explored. And the explorer will be wise if he not merely follows the continuous march of the main argument, but makes it his special business to examine those pregnant germs of thought which he will there find scattered in such abundant measure.

THE PROBLEMS OF LOGIC

BY

FEDERIGO ENRIQUES.

INTRODUCTION.

A STUDY of the meaning of Logic and its special contributions rendered to thought, as it developed within the boundaries of Greek philosophy to its consummation in the Aristotelian system reveals to us three different ways of looking at things, superimposed one upon another:

1. A *metaphysic*, which strove to arrange according to genus and species the relations it found obtaining within the actual world, and, by means of the classification thus procured, to lay the foundation of *Science.*

2. A *theory of concepts*, which concerned itself directly with the products of human thought, and sought to discover the laws by which the latter is governed in its work of combining the simpler elements and building them up into definite structures.

3. A doctrine which endeavoured, by an analysis of *verbal forms*, to determine expressions exactly corresponding to the laws which govern the understanding in its pursuit after knowledge.

The metaphysical Realism worked out by Plato and his followers combined a metaphysic of the actual with a theory of the concept, and held itself aloof from all Sophistical argumentation. On the other hand, as might have been expected, the analysis of thought was confounded with the inquiry into its verbal expression, and Logic may be said, to a certain extent, to have merged into Grammar.

This formalistic interpretation of Logic carried all before it in the *Scholastic* Philosophy, and finally resolved itself, by way

of Nominalism, into a dry working out and grouping together of verbal schemata.

Meanwhile, various tendencies combined to bring about a scientific re-birth of Logic as a realistic doctrine. Indeed, some of the latest representatives of Positivism, in their reaction against metaphysical Rationalism, have gone so far as to deny altogether the value of the Aristotelian Deductive Logic, and have substituted for it an Inductive Logic, in which a theory of experience is opposed to the old theory of reason.

Exaggerations in this direction, however, have never been able to secure a foothold within the sphere of Mathematical Philosophy, which must be regarded as the rightful heir of the great speculative tradition.

But only within quite recent times have mathematical philosophers resumed the study of logical problems.

A new school of "Logistic" or "Mathematical Logic" has arisen. Its special object is the revision of the Aristotelian schemata; or, in other words, the *analytical study* of the different *modes* in which *thought finds expression* with a view to substituting a *system of symbols* for the old and inexact expression in words.

Yet another line of investigation has led to developments, parallel to but independent of the foregoing, which we will here briefly set down.

Logic is here regarded as the *theory* of those *mental processes* which we denote by the term "rational," and is pursued from this point of view alone, apart from all considerations as to the particular medium of expression of thought by means of words or signs. It finds its object of investigation chiefly in the developments of abstract Science. Its method is as follows: adopting psychologically an ejective point of view, it proceeds to search out and compare, to analyse and correct the different conceptual structures, and the marks by which, in an evaluation, they are recognized as valid. In other words, a special inquiry is made into the value of Logic considered in its relations to reality, and its consequent applicability to knowledge; this inquiry maps out the leading principles of a Theory of Knowledge which may be called *Critical Positivism*. The philosophical attitude involved is that of a *Scientific Rationalism*, which strives to reconcile certain traditional oppositions dating

back to Descartes and Bacon by means of a deeper criticism, suggested by some of the fundamental views of Kant.

I. LOGIC AS THE SCIENCE OF EXACT THINKING.

OBJECTS. Logical thought occupies itself with *objects* which ordinary speech accepts as given. But what exactly is an "object"? It is essential for the interests of Logic that this general idea should be clearly defined. With this end in view, we offer the following observations:

1. Thought takes for its objects immediate data of experience, namely, *things* from the perceptible world, such as stone, house, animal, etc. But, in defining or drawing inferences from these things, it always presupposes certain conditions, in virtue of which these data are taken as "objects of thought." It assumes, for instance, certain conditions of physical unchangeableness, so that experience can offer us *equivalent percepts*, which can be related in thought to one and the same object. A crushed stone, a fallen house, a dead animal, are no longer the same objects as those just now denoted by the terms: stone, house, animal.

2. Thought takes as its object *abstracta from perceptible things* or from the *relations* between them. The *implication* here is of a something that is common to all the different perceptible things, or groups of perceptible things, which can be arranged under a certain *classification*. For instance, the abstracta "horse," "dog," "man," etc., being taken as objects, a classification of living creatures is then accepted as given—*e.g.* every creature belongs to a class or kind, and to one only. The abstract concept of living creature, taken as the object of thought, corresponds in the same way to all the individuals belonging to one and the same class, which can be regarded as *equally representative* of the class.

In the case in which a "relation between things" is taken as object, the quality common to all these things is considered in abstraction from the things themselves. Thus, for example, we speak of geometrical coincidence or equality: the idea here before us is that of all possible pairs of coincident figures; and the relation of "coincidence" is abstracted from these pairs, and is then thought of as a something common, which a close examination of them has revealed to us.

3. Thought takes as its object a *thing* or a *relation* as merely *possible*, independently of whether or not it actually exists. Here the object is the immediate product of the thought which posits it.

4. An indeterminate object may be taken as such hypothetically by means of a sign or word which stands for it. In this case, the sign or word must vary in meaning according as it stands for different objects. On the other hand, different equivalent signs (symbols or words) may be attached to one and the same object; thus, for instance, the signs a, b, $c \ldots a'$, b', c', may be so arranged with respect to the objects denoted respectively by A and A' that every sign belongs to one class, and to one class only, since every class is composed of equivalent signs which correspond to the same object. Thus,

$$\begin{array}{ll} \text{Corresponding to } A & a = b = c \ldots \\ \text{Corresponding to } A' & a' = b' = c' \ldots \end{array}$$
$$a =|= a'$$

LOGICAL PRINCIPLES. If we now compare the different cases in which thought posits an "object," we find that they all possess the following characteristic: An object is always compared by thought to some *datum* and recognized as similar to it whenever the likeness is such as to enable the presence of a common element in the different presentations to be detected; the presence of this common element is then taken as a sign that an object which has already been thought is being thought over again. More briefly, a *logical object* is something which is *posited as an invariant of thought.* Such an invariability implies conditions, and these conditions are given in the principles of Logic, taken in their most elementary sense.

1. The principle of Identity.
2. The principle of Contradiction.
3. The principle of Excluded Middle.

The import of these three principles is as follows: When two objects are thought simultaneously a judgment is passed as to their identity or difference. This judgment is determined entirely by the objects themselves; the moment at which they are thought, and the fact that the thought is or is not a repetition of a preceding one, are not taken into consideration.

These principles, then, co-operate in transforming a temporal

and genetic process of thought into an intellectual system of contemporaneous relations; they give us, as it were, a momentary vision of the *timeless*. We get an example of such transformation in the representation of the above-mentioned process by means of signs or written words. In this connexion we may say that the principles of Logic express the conditions of the possibility of the expression of thought in speech-schemata, or in a system of signs. It follows from this that since the laws are themselves implied in the use of signs they cannot be deduced from any special relations between signs; hence the relation $a=a$ does not express the principle of Identity but is a mere tautologous judgment which states nothing.

NOTE. If we refuse to recognize in the principles of Logic the conditions of connected thinking (definition of "object"), and regard them instead as judgments relating to the actual, we shall be forced to join company with the Eleatics and to deny the flux of perceptible things. For when related to Reality at a particular instant these principles lose all meaning. Hence, if taken as objective principles (cf. II.), the principles of Logic become either false or futile.

LOGICAL OPERATIONS. The logical process of thought in which objects that have already been identified or distinguished are taken as data consists of various operations, by means of which it is possible, starting from the given, to create (define) new objects.

Such fundamental logical operations are:

1. The *combining* of several objects in a *class* (group, totality), or even the *arranging* of various objects in a *series* (subordinate class).[1]

2. The positing of a correspondence between two classes or series, such that with every element (object) of one a certain homologous element of the other is associated (mathematical concept of *function*).

To these operations, which can be shown to be simple cases of *psychological association*, are opposed those which divide (dissociation).

3. The *distinguishing* of two or more classes, that is to

[1] This latter operation can be reduced to the former if we assume classes to be so formed that every element in turn can be taken together with the succeeding one. In this way, Symbolic Logic seeks to avoid a reference to the genetic character of thought.

say, the determining of the common elements collectively (*Interferenz*).

4. The *inverting* of a given function or a given correspondence between two classes (*Inversion*). Generally speaking, in such cases, *several* elements of the first class correspond to every element of the second class.

We have in the process of *abstraction* an especially remarkable instance of this operation. A class (a, b, c ...) can always be thought as a single object A, which corresponds to the elements a, b, c ... (that is to say, is a function of the said elements). If, then, we wish to invert the correspondence, we must pass from A to each of the objects a, b, c ... which are thought as *interchangeable* (*equal*).

To think a as different from b, c ... and to think it as similar to b, c ... is in reality to think different objects; the first is the object a, which is taken as given; the second is the *abstractum a of the class A.*

The abstractum of the element of a class, then, becomes in its turn a new object, by means of which we can regard the elements of the same class as interchangeable presentations, since their differences are ignored by thought.

EXAMPLE. Titus, Gaius, Sempronius are human. Each one of them corresponds to a representation of the abstractum "man."

CONCEPTS. An object which can be interpreted as "the abstractum of the element of a class" forms a *concept*, and is created or exhaustively explained by the abstraction to which it owes its existence. For example, after the persons Titus, Gaius ... who constitute the members of a certain society have been named, the concept "member of this society" may be said to be determined by abstraction. But in addition to these concepts which admit of exhaustive description, thought also accepts as data concepts as to the meaning of which we subjoin the following remarks:

1. A *given concept* may correspond to a *class of posited objects* when the latter can be thought as *determinations peculiar to it.*

2. The concept is thought as the abstractum of all the determinations of which it is capable, and is expressly explained as such if the corresponding determinations are actually thought: this is only possible if they form a *finite class.*

3. It is possible for a concept to admit of an *infinite* number of determinations; in that case it is impossible to actually think the objects involved: but

(*a*) We can decide if any particular object is or is not a determination of this concept.

(*b*) All the objects which are possible determinations of this concept are determined *a priori* as belonging to one and the same class. The concept, as it were, presupposes the law according to which thought can create an *unlimited number of determinate classes.* The statement of this law runs thus: the *possible determinations* of a concept form an *infinite class* of which in turn the concept itself is the abstractum.

EXAMPLES. The concept "circle" corresponds to a class of infinitely many circles, of which only a finite number can be actually thought. A figure being given, any one acquainted with the concept can form a judgment as to whether it is or is not a circle. And however many circles be given, they are determined in advance by the concept as members of one class; and this class becomes larger and larger in proportion as more and more circles are successively thought. The assumption of an infinite class of circles implies this pre-determination of an infinite series of finite classes, and the series embraces all the circles which have been thought at any given moment.

CONTENT AND COMPREHENSION. A concept *c* can be determined by assigning either

1. The *class C* of objects of which it may be regarded as the abstractum; or,

2. The *sum-total of the classes* which contain *C*, and which are not identical with it.

The class *C* is called the *content* of the concept *c*; the sum-total of the classes which contain *C* form the *comprehension* of the concept itself.

Thus the concept can be determined either according to its comprehension or its content.

These definitions may be illustrated by a well-known geometrical figure. The characteristic determinations of a concept *c* are represented by points which lie within a circle *C*; the surface of the circle represents the content of the concept, and the totality of all the circles or of all the plane surfaces within closed lines which contain *C* represents its comprehension. Each one of these surfaces corresponds to a more

general concept which includes c, and which is reached by abstracting from one of the *characteristic marks* of c.

DEFINITION. The *exact definition* of a concept c is reached by means of other concepts or given objects $a, b \ldots$, whenever c is derived from $a, b \ldots$ by means of logical operations (combining, distinguishing, etc.) which can be expressly indicated.

The classical doctrine of definition was formulated mainly with reference to the cases in which the comprehension of a concept can be determined by the difference between more general concepts: this is the origin of the scholastic rule of definition by proximate genus and specific difference.

But we here have an opportunity of considering the theory of definition from a wider point of view. There are, or may be, as many kinds of definitions of concepts as there are, or may be, systems of logical operations. And there are *elementary types* of definition which correspond to elementary operations, thus:

1. Definitions *by combination*, in which the concept c is reached by combining a given $a, b \ldots$: for example, the perimeter of a polygon is arrived at by combining the length of the sides.

2. Definitions *by differentiation* (*Interferenz*). One and the same concept c may appear as the *difference* between certain given concepts $a, b \ldots$, and $a', b' \ldots$. A definition by differentiation is complete when the classes corresponding to $a, b \ldots$ (as compared with $a', b' \ldots$) can be differentiated as far as a class c which corresponds to a, and when they have no other element in common.

3. Definitions *by correspondence*. A correspondence between two classes of concepts (or objects) $a, b \ldots$ and $a', b' \ldots$ is taken as given; a concept can then be defined as that which corresponds to another *datum*, within the established correspondence.

EXAMPLE. The definitions in which familiar correspondences are called to mind by means of such phrases as: the father of ..., the king of ..., etc. An important example of this species of definition occurs in Natural Science when a phenomenon b is explained as *caused* by a. We must note here that this definition is generally unacceptable, since a phenomenon may be produced by several causes. At any rate this is so until the successive order of phenomena is held to be predetermined, when causality will cease to play an ambiguous part.

4. Definitions *by abstraction* (Grassmann, Helmholtz, Peano, etc.). These are the definitions which led us to investigate the significance of the concept.

An object is defined by abstraction when it can be shown to be a member of a class; the relation then appears like an equation. This relation must exhibit three fundamental properties, called by Peano:

(*a*) Transitive property; if $a = b$, and $b = c$, $a = c$.
(*b*) Symmetrical property; if $a = b$, $b = a$.
(*c*) Reflective property; $a = a$.

EXAMPLE. Euclid was the first to give a definition by abstraction which could be used for scientific purposes when he defined the relation between two magnitudes by comparing them with a similar relation (Proportion).

The direction of a straight line is also defined by abstraction when it is defined by means of the relation between two parallel lines which possesses the three fundamental properties of an equation.

Hence the most modern physicists define their fundamental concepts by abstraction; thus, for example, Maxwell and Mach define "mass" by the relation "equal masses," the physical content of which they explain.

In Political Economy, also, the concept of "value" expresses an abstractum; namely, that of equivalent (*i.e.* exchangeable in commerce) goods.

PURELY LOGICAL CONCEPTS. Logical operations start from given concepts, and their aim is to create new ones. By abstracting from the differences which reside within their meaning purely logical concepts are reached; these are used as *general schemata* for all the concepts which have been created by scientific or ordinary thought and for all the relations which hold good between them.

Such concepts are *class, totality, series, agreement*, etc., whose names we have already encountered.

All the conceptual relations which are logical in character can be traced back to the relation of "*belonging-to*" between an element and a class, or to the *inclusion* of one class in another.

The relation of *equality* is the most important of those so derived.

As a matter of fact, the first form under which we encoun-

tered the operation of equating was that of the interchange of the objects of thought with the abstractum corresponding to the class in which the objects were combined. This point of view gives us the first of the following definitions of the concept of equation :

1. The equation $a = b$ expresses the fact that a, b are members of a class the abstract concept of which has been created by thought.

A second way of defining equation is by examining the conditions under which a relation between a, b, c can be reduced to the fact that a, b, c are members of a class. This inquiry, which we made in the preceding paragraph, showed us that

2. Equation is a relation which exhibits the transitive, symmetrical and reflective properties.

Lastly, we may investigate the significance of equating from a third point of view. If we write $a = b$ we posit a relation between two different objects, but they are thought as combined together in the same class, to which class a single abstractum corresponds. Hence the above-named relation may be explained by writing: abstractum $(a) \equiv$ abstractum (b), the sign $\equiv$ standing for "identical with."

Thus every relation of equality is reducible to the identity of an abstract concept, which can be conceived as explained or produced in two different ways.

It is from this standpoint that A. Padoa gives his definition.

3. An equation implies the identity in extension of abstract concepts, which are definable in different ways as regards their comprehension.

EXAMPLE. Equilateral triangle = equi-angular triangle. One and the same class of equilateral and equiangular triangles is defined by means of two different characteristic marks: once as the differentiation of the class "equilateral polygons" from triangles, the second time as the differentiation of the class "equiangular polygons" from triangles.

We note that this way of defining equation, differentia $(a, b \ldots)$ = differentia $(a', b' \ldots)$ is really the same as definition 1, for if we regard the differentia $(a, b \ldots)$ as a function of the classes a, b then the content of the "class differentia" is formed as an abstractum from all the self-differentiating classes.

LOGICAL RELATIONS. If a concept c is formed from other given concepts a, b ... by means of logical operations, and can consequently be expressly defined as originating in them, the logical operations by which c was constructed are conceived as *logical relations*, obtaining between c, a, b ....

But whenever two concepts c, d can be defined by means of other *data* a, b ..., some kind of logical relation, as a general rule, exists between c, d. For instance, a logical relation obtains between c, d, if c includes d, if c and d are differentiated concepts, etc.

In ordinary speech logical relations are expressed by *propositions.*

The singular proposition "A is B" and the universal "every A is B" express the relation of inclusion; the concept B contains A; in other words, the class corresponding to the former (B), contains the class corresponding to the latter (or the individual). While the particular proposition "some A is B" denotes, on the other hand, that the two classes corresponding respectively to the concepts A and B are different from one another (that is to say, have common elements).

It is important for us here to gain a clear understanding of the (psychological) import of the logical relations between concepts.

As a general rule, logical relations imply logical operations (such as combining, distinguishing, etc.), *the work of which on the concepts and other data involved is taken as done.* There are, however, two ways of regarding logical relations:

1. *As Created Relations.* When, as the result of operating on certain data a, b ... a logical relation between the concepts c, d ... is posited, by which they are defined; for example, the relation between $a+b$ and a.

2. *As Given Relations.* When immediately given concepts are combined and the possibility of creating new ones on the basis of other *data* is abstractly expressed.

Let us take, as an example, the proposition: "If a *straight line* has two *points* in common with a surface, it belongs to the *plane.*"

We have here three concepts: point, straight line and plane. The straight line and the plane are conceived as "classes of points." The relation expressed implies a condition, and the assumed combination of the infinitely many points into the

classes denoted by "straight line" and "plane" must satisfy this condition. There exists, that is to say, a *given relation* between the concepts "straight line" and "plane."

DEDUCTION. Two logical relations or systems of logical relations S, S' are *equivalent* when they are or can be established by the same logical operations; or, in other words, when the conditions governing the constructive operations which they express are the same in both cases.

The system S' *may be inferred from* or is the consequent of S whenever the conditions implied in the construction of S' are such as are realized in that system of constructive operations which satisfies the conditions implied in the construction of S. When S' follows from S, and S from S', the two systems are equivalent.

When we talk of deducing new logical relations from a given system S, we mean that we exhibit relations which, collectively and in detail, are inferences from S.

The possibility of such a deduction is contained in the fact that every operation or every system of logical operations can be transformed into various equivalent systems of relations. For example, if the classes a, b, c are combined into (abc), we can arrive at the same result by combining the classes (ab) and c. Thus the relations

a belongs to (abc)
b belongs to (abc)
c belongs to (abc)

can be combined as in the following system:

(ab) belongs to (abc)
c belongs to (abc)

and the latter is equivalent to the former.

AXIOMS. Among the purely logical concepts which form the general schemata of the cognitive concepts there are some *universal* relations which express the fundamental properties of all logical operations, and form the elementary principles of deduction. Such relations are called *axioms*.

Examples:

1. The *associative property* exhibited in the operation of combining: The class (abc) is the combination of (ab) and c.

2. The *commutative property* of the same operation: If

several concepts a, b ... are arranged in series which are regarded as equal, however much they may differ in kind, the abstract concept of the series (ab) is the class (ab ...).

3. The *transitive*, *symmetrical* and *reflective properties* of equation.

4. The *principle of inclusion*: If the class a includes b, and b includes c, then a includes c, etc.

This principle appears as the *principle of substitution* in the theory of inference: If a is b (the concept b being the abstractum of the other concepts combined wtth a), and if b is c it follows that a is c.

We must notice here a *logical contradiction* or paradox which lurks in the verbal expression of logical relations. The paradox consists in the formal application of the rule to a single case, as in the following:

The apostles are 12 in number.
Peter and Paul are apostles.
∴ Peter and Paul are 12 in number.

Peano meets the difficulty by distinguishing two different meanings of the copula, which he symbolises respectively by $<$ and ε.

This device answers the purpose for which it was invented, but it fails, in our opinion, to explain the paradox. In our view it is not the copula but the middle term that changes its meaning. In the major premiss it refers to "the class of apostles" (which is said to consist of twelve), while in the minor premiss "apostle" is the abstract conceptum of an individual of the class. Peano was obliged to distinguish two meanings of the copula, because he omitted to differentiate between a class and the abstract concept immanent within it.

DEDUCTIVE THEORIES. The foregoing considerations will help us to understand the structure of a deductive theory. Let us take some deductive theory half-way through, omitting all inquiries as to the principles implied therein: for instance, any book on Geometry *after* the first chapter. We shall find that in the development of the theorems *certain concepts, with the logical relations obtaining between them, are accepted as given.*

The development consists in the express declaration of new concepts which are produced out of those originally given by

means of logical operations, and in the *deduction* of new relations (*theorems*) from the given relations.

Let us now try to re-ascend, as far as we can, the ladder of explanations and deductions which constitutes the theory. Since both explanation and deduction point backwards we must, at the beginning of the theory, necessarily find:

1. *Fundamental or primary concepts*, which are not themselves explained, but by means of which all the concepts occurring in the theory are explained.

2. *Fundamental or primary relations* (axioms or postulates) which are not themselves deduced, but from which all the theorems of the theory are deduced.

There is a certain *freedom of choice* in selecting which concepts and relations shall be taken as primary in any particular theory.

For in place of a system of concepts $A, B, C \ldots$, and of relations $a, b, c \ldots$, another system of concepts and relations $A', B', C' \ldots$ and $a', b', c' \ldots$ can be substituted as long as it is *equivalent* to the original one. Equivalence, in this connexion, means that $A', B', C' \ldots$ can be completely explained by means of $A, B, C \ldots$ and $a', b', c' \ldots$ deduced from $a, b, c \ldots$, and the reverse.

The postulates of a theory are said to be *independent* when no system deduced from less than all of them is equivalent to the one deduced from all of them; in other words, when no one of the postulates can be deduced from any of the others.

The fundamental concepts of a theory are indeducible when none of them can be explained by means of the others, even with the help of those fundamental relations which were assumed as obtaining between them.

This question as to the independence of the principles and the indeducibility of the concepts of a deductive theory has given rise to remarkable developments within the sphere of mathematical inquiry.[1]

IMPLICIT DEFINITION. We subjoin some remarks on the points which are of most importance from the philosophical point of view.

[1] Cf. F. Enriques, *Problemi della Scienza*, 2nd edition. Bologna, 1910: Zanichelli. (German trans. Leipzig, 1910: Teubner.) *Questioni riguardanti la geometria elementare*. Bologna, 1900: Zanichelli. (German trans. Leipzig: Teubner.) "Prinzipien der Geometrie," *Encyklopadie der mathem. Wissensch.* Leipzig, 1909: Teubner.

There is one specially remarkable case in which the concepts and postulates of a deductive theory are the more completely explained the more rigorously the analysis of its axioms can be carried out. This is the case whenever all the concepts can be adequately explained by means of given objects, not mutually related (theory without postulates), in proportion as the relations developed in the course of the theory are generated by the same operations as those that explain the concept. Let us assume, for example, three classes, A, B, C, and the three following relations existing between them:

A	and	B	have	an element	c	in common.
A	"	C	"	"	b	"
B	"	C	"	"	a	"

In such a case, we may assume that the objects a, b, c are not mutually related, and we arrive at the following explanation:

$$A = (bc),\ B = (ac),\ C = (ab).$$

Our supposititious case corresponds to certain deductive theories in which the concepts presuppose a finite number of objects from which they are generated by means of logical operations. Thus the whole theory is an actual logical progress.

But, as a general rule, this is not possible. The geometrical postulates, for example, which combine the concepts of point, straight line and plane, assume that the straight line and the plane include an *infinite number* of points. Hence the relations posited as existing between the concepts express the conditions of a logical process which can never be imagined as completed.

But we can never assume as given objects by means of which all the concepts of a theory can be exhaustively explained. We must rather regard the *necessarily assumed postulates* as *implicit definitions of the fundamental concepts*. The question then arises as to whether the logical process demanded by the postulates is *possible*; that is to say, whether the postulates themselves can be accepted. This question can only be answered by *existential hypotheses* based on immediate intuition or on physical, psychological or historical experience: exactly as has been done in the different critical discussions as to the foundations of Mathematics.

II. LOGIC AND REALITY.

LOGIC AND METAPHYSICS.—VALIDITY OF THE LOGICAL PRINCIPLES. The processes of logical thinking enter into practical life, and are still more indispensable in the building up of the sciences; for only by their means can ideas be rendered determinate. This use of Logic as an instrument of knowledge presupposes a certain correspondence between thought and reality, or, at any rate, between the laws which govern the one and the other. Hence we are led at once to the question which is *fundamental for knowledge*: how is the application of Logic possible?

Two points which are intimately related call for special investigation:

1. As to whether and how the immutability of logical objects implies the persistence of a corresponding something in the world of Reality.

2. Whether there is anything in Reality corresponding to general and abstract concepts.

With regard to the first question, we find four different attitudes have been taken up at various times by philosophical thought.

THE DOCTRINE OF THE ELEATICS. The principles of Logic imply the immutability of everything that is real; all that is is rational: hence change and motion have no real existence.

THE DOCTRINE OF HERACLITUS. Reality is a flux of sensuous things: hence nothing in it is rational. (Impossibility of the application of the logical principles.)

THE DOCTRINE OF HEGEL. Reason must overcome the contradiction between understanding, which constructs for itself unchangeable objects (by means of logical thinking), and empirical reality, in which everything is changeable. Hence we must create a *higher Logic* to supply the theory of the thought which transcends experience. Reality will be seen to be rational when it is looked at from the standpoint of this so-called higher Logic (*Dialectic*).

DOCTRINE OF CRITICAL POSITIVISM. Critical Positivism regards the system of Hegel as an unsuccessful attempt to base scientific procedure upon a correspondence between the understanding and experience. If we are to have a rational synthesis of knowledge, it must no more violate the laws of logical

thinking (intellect, understanding) than it must ignore the claims of the empirically given. Hence Critical Positivism does not deny the validity of Logic, but makes the following admissions:

(*a*) That the objects of logical thought are unchangeable.

(*b*) That Reality itself is changeable, but that within the changing flux things and relations can be distinguished which change so slowly that they may be treated as logical objects.

(*c*) That the approximate correspondences established between logical objects and unchanging realities admit of progressive correction by the help of deduction and experimental verification, and that in this way new and stricter immutabilities can be arrived at.

Thus to synthesize understanding and reality is to progress towards an exact selection from given realities of those which satisfy the conditions laid down by the logical principles: *Reality, that is to say, is already partly rational, and science strives to make it progressively so.*

VALUE OF THE AXIOMS. The possibility of progressively rationalizing Reality, which is implied in scientific construction (justification of deductive theories), is based upon a fundamental demand, and this demand constitutes the objective value of Logic.

As already said, the principles of Logic make no assertion as to the nature of the real, but simply state the conditions which objects of thought must satisfy. To this we must add: when these conditions are satisfied, those axioms which state the laws of logical operations and the fundamental properties of the classes created by thought also sum up the properties of the classes or organisms which correspond to these in reality.

Subject to the conditions of unchangeableness expressed by the principles of Logic, the totality of real things exhibits the properties demanded by the axioms (combining, changing, etc.).

Thus, for example, cashiers test the real value of the logical developments of arithmetic when they compare their calculations with actually existing currency.

Hence we may say in general terms that the constancy of the objects or relations which form the subject-matter of a scientific theory denote the *limit* to which previous thinking had

carried the *application of Logic*, without which there can be no theory at all.

REALISM AND NOMINALISM. But Critical Positivism cannot halt here; it must discuss the deeper problem of general ideas. Are there realities corresponding to ideas or general concepts? This is the question which gave rise to the famous scholastic controversy between the *Realists* and the *Nominalists*. We will here briefly set down their most characteristic doctrines. *Realism* runs through the whole series of concepts, passing from the general to the particular. The concept of genus (class) or type exists in the mind prior to that of the individual: this last is nothing but the sum of a common element (the abstractum) and of a particular element (characteristic mark of the individual). Reality is regarded as corresponding to thought in such a way that conceptual relations may be taken as copies of real relations: hence general ideas are believed to reflect the *essence* of things; every individual object exhibits marks peculiar to itself in addition to those of its species; every species possesses, in addition to the characteristics of the genus, those in virtue of which it was "differentiated" from the genus (specific difference).

REALISM is open to *criticism* from two points of view:

1. If by reality is meant the world of experience, we find in the latter concrete individuals only, never abstractions. "This horse" exists, but not "horse-ness," nor yet the "horse-type."

2. From the psychological point of view (taken up by Berkeley, Hume, Stuart Mill), it is not true that the general idea precedes the particular. On the contrary, the genetic process of thought always works inductively from the particular to the general. Moreover, abstract ideas are not formed on the model of particular ideas, but are derived from the interchangeability of several associated ideas.

NOMINALISM, which is based on the foregoing criticism, arrives at the following one-sided interpretation: Reality consists of a plurality of separate objects which are not mutually related. Their combination, and consequently the formation of concepts, is the result of an *arbitrary* process of thought, to which nothing in Reality corresponds. Hence, as knowledge, Logic possesses no value.

The point in the Nominalist theory against which Empirical Philosophy directs its criticism is the tearing apart of

Reality into separate objects: going back to Heraclitus' doctrine of the inter-connectedness of Reality and the flux of perceptible things, *radical Empiricism* may be looked on as a *completion* rather than as an annulling of Nominalism; although from other points of view this sphere might seem adapted to the revival of certain mystical forms of Realism.

CRITICAL SOLUTION OF THE PROBLEM OF GENERAL IDEAS. The critical point of view admits:

1. The inter-connectedness of experience (the demand of radical Empiricism).

2. The psychological explanation of the formation of concepts, as put forward by the English Nominalists. But it claims

3. That within the flux of Reality approximate invariables can be distinguished (objects).

4. That while the combination, in accordance with logical principles, of such objects into classes is an *arbitrary* proceeding on the part of thought, yet those combinations which are made in the interests of knowledge are determined by certain *purposes* (motives) which invest the concepts so made with real significance, and which express themselves by positing objects which satisfy the logical conditions as regards *similarity*.

5. The creation of scientific concepts, thus determined, leads us to the recognition of something *more unchangeable* than the objects from which we started.

The theory here sketched may be termed *scientific Realism*. It recognizes in the logical advance of science *an uninterrupted, but progressive series of mental constructions, which series gives us an approximate idea (representation) of the inter-connected system of Reality.*

We subjoin a few remarks in elucidation of this theory.

It is true that in the world of experience we never meet with the "lion-type," but only with actual lions. But since Plato admitted the existence of "lion" in the world of ideas, the empirical interpretation of Realism was from the beginning excluded. What did Plato really mean by the mental (ideal) existence of the lion as the type of its species? It appears conceivable that he regarded the type as an ideal norm, to which concrete individuals approximate. In this norm all the relations which appear in individuals in varying degree are reflected in harmonious simplicity.

So interpreted, Platonic Realism may be brought under the

point of view of modern Natural Science, which ascribes a real value to the species (biological or mineralogical) in so far as certain averages tend to exhibit a greater regularity and persistence in the appearance in individuals of the characteristic marks. The only important difference is that:

1. Platonic Realism lays great weight on the invariability of species, and by means of this idea constructs a Metaphysic which does not lend itself to the modern evolutionary theory.

2. Scientific Realism, however, cannot repeat the error contained in the Eleatic theory, and ascribe to Reality the strict invariability demanded by the concept. The invariability of species is admitted as an approximate hypothesis only, requiring confirmation by *a posteriori* evidence.

In the same way, with regard to the concept of *physical* or *natural law*, Scientific Realism adopts the attitude which was introduced into modern science by Galileo and Descartes. The science of Mechanics, for example, examines the relations by which the motions of bodies are determined not under actually existing but under artificially simplified conditions, and therefore arrives at abstract results. The *real* value of its conceptions is measured by the degree of approximation to truth with which they enable facts to be predicted. But it cannot be denied that closer approximations to invariability are discovered by scientific procedure than any which we encounter in ordinary life. For example, the value of the constant g for gravity is much more certain than the roughly calculated acceleration of a falling body, in which the necessary corrections are not allowed for.

SUBSTANCE AND CAUSE. But we must go deeper than Scientific Realism does, and test the value of the two fundamental categories of relation which are expressed in the laws of Nature. I mean those of *Substance* and *Cause*.

The concept "Substance" implies an order of phenomena permanently existing together; it is assumed that certain perceptible characteristics combine into definite groups, whose persistence is then conceived as forming the substance of perceptible objects. Thus, for example, "energy," "matter," and even the different kinds of matter which constitute the so-called simple bodies, are scientific concepts, falling under the category of substance.

Scientific inquiry always employs the categories of Substance

and of Cause whenever there is an *a priori* possibility of discovering relations which come under these categories. Scientific results confirm, though only approximately, the constructive hypothesis. In this way the significance and value of the hypothesis itself is brought to light, especially when, with the progress of science, a new understanding and application of it becomes necessary. This is how we have arrived at the truth contained in the hypothesis mentioned above:—*In the world of perception there exists a mutual dependence, or reciprocal relation, which binds together all objects and all appearances to those which are proximate to them in either space or time. The laws which affirm a relation of Substance or of Cause imply a selection of objects or conspicuous phenomena, a change in which brings with it a noticeable change in the data. Hence the lack of a law for simultaneous or serial phenomena must be made up for by approximating the phenomena involved as closely as possible, and by paying stricter attention to the relations which were neglected when the law was formulated.*

But if, instead of accepting an approximate result, we insist that *all* the relations of Substance or Cause of which any given law is supposed to supply corrections must be taken into consideration, we shall find ourselves confronted with the difficulty of ascribing any meaning at all, in a transcendental sense, to any definitive expression. For the universe consists of an *infinite number* of related data, acting and reacting upon one another. It is precisely in this circumstance—in the fact that scientific Realism admits of an unlimited advance in a graduated perfection—that we find the essential difference between it and metaphysical Realism.

THE PROBLEM OF METHODOLOGY. So far we have been occupied with the critical inquiry as to how those general principles are to be discovered by which alone the application of Logic to reality, and, as a consequence, the construction of a rational science, is possible. But when we reach the methodological point of view we find that the theory of knowledge contains an element of Pragmatism.

Stated in general terms, the problem of Methodology is to determine how the acquirement and extension of scientific knowledge takes place.

The philosophical attitude we have taken up permits us to adopt the proposal of Stanley Jevons and Claude Bernard, and

exhibit this process under a schema which is merely a modification or extension of the schema already in use for inductive reasoning.

Jevons has the great merit of having reconciled the opposition between the inductive and deductive sciences; for he recognized the function which deduction fills as an instrument in inductive proof. He distinguished four different stages in induction :

(1) Pre-scientific observation ; (2) hypothesis ; (3) deduction ; (4) verification.

The schema actually employed in scientific procedure is much more complicated than this simplified one of Jevons ; for every inquiry presupposes that some knowledge has already been acquired ; while the comparison of observations and experiments made in the light of a theory previously held often suggests the hypothesis ; not of course expressly, but by means of concepts which have been formed in the course of the inquiry.

We must therefore modify the inductive schema as follows :

(1) Pre-scientific observations and experiences (negating or affirming concepts hitherto employed) ; (2) construction of concepts which hypothetically exhibit the nature of the real ; (3) deduction ; (4) verification.

This last stage, in so far as it tends to confirm the hypothesis, provides a starting-point for a further inductive process. Attention must be drawn to the part played by hypothesis, according to this schema, in the conceptual representations of a group of phenomena which together constitute a scientific inquiry. Every theory contains :

1. *Implicit* hypotheses, involved in the positing of certain data or relations as objects of logical thought (such hypotheses, for instance, as are implied when we speak of a "mass" or employ the constants (invariables) of Physics.

2. *Explicit* hypotheses, which, by the help of deduction, can be brought to the test of verification by experiment. But here we must notice carefully two points :

(*a*) Under artificially simplified conditions, the result of experiment may offer an affirmative or negative answer to the question formulated in the hypothesis, *when the implicit hypotheses have been correctly apprehended.*

(*b*) More often, however, such results confirm or refute not

a single hypothesis, but an inference which follows from a *system of hypotheses.*

It may happen, therefore, that the inferences from any given system are partly confirmed and partly negated; in that case those which have been confirmed are taken as the starting-point for further inductive processes, leading to more general hypotheses.

The value of generalizing by inference comes out here very clearly.

Implicit hypotheses must be regarded as the least variable element in our systems of knowledge, for they are not contained in the postulates of any particular theory, but are rather immanent in an entire science. Nevertheless, the revision of these hypotheses from time to time is an imperative duty. *The confirmation of such implicit hypotheses does not depend on the success or failure of definite experiments, but upon the congruence of experimental results as a whole: the criterion of implicit hypotheses is the Axiom of Contradiction.*

PRESUPPOSITIONS OF EXPERIMENTAL PROCEDURE. The points of view from which we can estimate the probable worth of experiments tending to confirm given hypotheses and a given scientific theory call for a special methodological inquiry.

One important result of such an inquiry is that it brings out the fact that the value of experience in general is dependent not only on the implicit hypotheses contained in the science under discussion, but also on certain presuppositions of experimental procedure which imply hypotheses as to the *continuity of the causal relation* or the mutual dependence of phenomena.

It is, for instance, by means of similar hypotheses, admitted *a priori*, that the science of Physics attempts to show that its phenomena are determined by forces *impinging upon one another in space and time.*

The presuppositions contained in the metaphysical notions of Substance and Cause also admit of further analysis; thus the causal relation can be traced back to a Group of *primary causes*, that is, to a constant transformation of a preceding causal into a consequent relation.

CONCLUSION.

It has been impossible to attempt, in this brief sketch, any exhaustive discussion of the problems touched upon; we shall be content if we have indicated their scope and their philosophical significance.

A more penetrating insight into these problems must necessarily wait upon the further developments of scientific procedure. This does not mean that the Theory of Knowledge expects illumination from the advance of experiment in the same sense as does, let us say, the science of Physics. Epistemological experience—if I may be allowed the expression—contributes not to Nature or to the subject-matter of Science, but to Science itself, regarded as a process of gradual conquest, and, in the last resort, *to human reason*, the instrument by which this process is carried out. Given reality, on the one hand, and the spiritual elements on the other, await one another. It is the task of human reason to draw them together into a synthesis.

In this sense we may say that the aim of Critical Positivism, as distinguished from previous attempts, is to carry out the programme laid down in Kant's *Critique*, especially within the spheres of Epistemology, Logic, and Methodology, in a scientific spirit consonant with that in which it originated.

THE TRANSFORMATION OF THE CONCEPT OF CONSCIOUSNESS IN MODERN EPISTEMOLOGY AND ITS BEARING ON LOGIC

BY

NICOLAJ LOSSKIJ.

I. THE STRUCTURE OF CONSCIOUSNESS AND OF KNOWLEDGE.

THE concept which plays the leading part in modern Philosophy is that of consciousness. As a general rule, the "immediate data of consciousness" form the starting-point from which modern philosophers proceed to build up their conception of the world. Epistemology, in particular, bases itself on an analysis of consciousness, on an investigation of the forms of consciousness, and so on.

Hence no modification of this concept can be safely neglected by the philosophical sciences, and no presuppositions must be allowed to lurk unchallenged within it.

The most dangerous of such presuppositions is that to which psychologism, subjectivism, solipsism, anthropocentricism and other similar theories owe their existence. The assumption is that consciousness is identical with the sum total of the *psychical* states of the *individual*, or, more exactly, with the *sum* of psychical states of which the individual *is aware*. If we start with this conception of consciousness, we must inevitably end in Pan-psychism, psychological Idealism, intellectualistic Phenomenalism, or some similar theory.

Owing to the wide vogue of this view of consciousness, and of the supposed evidence in its favour, an erroneous idea has sprung up that any Philosophy which takes the facts of consciousness as its starting-point must necessarily lead to

some such result. Some thinkers are content that it should be so; others, though not satisfied, can find no alternative. Among the latter are to be found those philosophers whose natural bent is towards Materialism. Under the influence of epistemological criticism, they believe it to have been proved that matter is only an "idea" ("idea" being taken in the sense that its *content* as well as its form is a psychical state of the subject knowing). Hence, though grinding their teeth with rage, they have to abandon not only their Materialistic theories, but also their belief in the existence of matter, as a non-psychical reality. There are others again who, recognizing the erroneousness of all attempts to reduce the world to an exclusively psychical process, feel it their duty to initiate a revolt against the habit which has crept into Philosophy of taking the facts of consciousness as a starting-point. Among these is R. Avenarius.

In the preface to his *Der menschliche Weltbegriff*, Avenarius says that there would be some sense in "starting from the facts of consciousness" if we had "an epistemological permit to explore the sphere embracing the human central nerve-organs in relation with their environment in order to start from it." But the entrance to this sphere is barred by the Idealists' discovery of "the immediate *data* of consciousness."

"It is true that where it is 'only' a question of *Psychology*, even a psychological Idealist is not so hard-ridden by his theories as never to appeal to conditions of the brain in 'explanation' of 'phenomena of consciousness.' But when it comes to *Epistemology*!"

"Fortunately I myself had long given up Epistemology in the scholastic sense of the word, so that I was prepared to be led by reflection to the *natural* starting-point of all inquiry. In my critique of pure experience I abandoned in favour of this the orthodox starting-point in the so-called 'immediate data of consciousness,' which is in reality the anything but 'certain' result of a *theory* still awaiting confirmation."[1]

Nevertheless, to start from the facts of consciousness does not necessarily imply a final surrender to psychological Idealism. Such a result is only inevitable if we base our investigation on the erroneous concept of consciousness to which we have already alluded. This, in its turn, depends upon a tendency,

[1] Avenarius, *Der menschliche Weltbegriff.* 2nd edition. S.X.

often unconscious, towards spiritualistic Substantialism or towards the mechanical and materialistic view of the psychical life. But modern Philosophy is gradually emancipating itself both from the old spiritualistic Substantialism and from the customary assumptions of the mechanical conception of the world. It is working out for itself an entirely new concept of consciousness, which is based not on unexamined conclusions from presuppositions dogmatically imposed, but upon *the exact description and articulation of the process of becoming conscious.*—It is remarkable that, amongst others, Avenarius himself helped to establish this new concept, and in his *Menschliche Weltbegriff* has even, all unawares, taken it as his starting-point. Hence his polemic against the habit of proceeding from the facts of consciousness is based entirely on a misunderstanding.

The new conception of consciousness which is leading Philosophy out of the *cul-de-sac* of psychological Idealism may be formulated as follows: Consciousness is the sum-total of everything which stands in a certain unique *relation* to the Ego. This relation is simple, and cannot therefore be described or defined by analysis into its elements. We can only indicate it by the following expression, which is metaphorical only, and must not be taken literally. Everything falls within the sphere of consciousness which the Ego "has." If we agree to call what the Ego "has" the *content of consciousness* (the expression is not quite free from objection, but unfortunately we can find no better one) we may formulate the theory of consciousness we wish to bring forward here as follows: Every fact of consciousness is made up of at least *three moments*; every such fact depends for its existence upon the presence of an Ego, of a content of consciousness, and of a relation between the two. According to this theory every fact in Reality with which *I am acquainted* is not merely a fact, it is also, owing to the circumstance that my Ego is connected with it by the relation of "having-in-consciousness," a content of consciousness; in other words, the Ego exercises towards it the *function* of becoming conscious.

The only necessity for consciousness is the presence of this relation or function. The nature of the content which enters into relation with the Ego is a matter of indifference. It is of no significance in this connexion whether the phenomenon we are dealing with is psychical or physical, whether it belongs to the

inner life of the subject or falls within the sphere of the *trans-subjective* world. Thus, to take an illustration, the motion of a pendulum is a non-psychical phenomenon, belonging to the trans-subjective world; nevertheless it may, by means of an act of perception, *become* part of the content of consciousness of a subject knowing.

The traditional theory of consciousness also recognizes a unique relation between the Ego and the content to be taken up into consciousness. The difference between the old and the new conception is that the new theory places in the foreground the *relation between* the Ego and the content of consciousness. It emphasizes *the functional character of consciousness.*[1] Hence it will not allow that the content of consciousness is psychical or subjective, but maintains this to be an erroneous assertion, based on dogmatic pre-suppositions.

This transformation, which appears at first sight so insignificant, leads to results of far-reaching importance. It seems, indeed, likely to lead Philosophy along entirely new paths. For the concept of consciousness is now seen to be far more comprehensive than was originally believed. A philosophy which starts from this concept is much more supple; innumerable new possibilities open before it. In proof of this we may point out that it has already freed itself with one blow alike from Solipsism and from the necessity of denying the existence of matter or of transmuting it by violence into a complex of psychical processes.

No one acquainted with modern philosophical literature can deny that this movement is widely extended. But, being still only in the early stages of its development, some confusions have arisen. Different philosophers employ different terms for what are in reality the same things; they choose different starting-points, and concentrate their attention not only on different points of the problem, but on points which lie far apart from one another. Then, too, there is still disagreement on essential points;—such, for instance, as to how the Ego is to be conceived, to what degree we are to assume that the content of consciousness is bound to the Ego by a constant bond, and so on.

Hence without a special investigation or a more or less exact analysis it is impossible to establish the affinity of the

[1] By the term "subjective content," as opposed to "trans-subjective content,' we mean the state of the individual himself, his joys, wishes, etc.

different theories of consciousness which have been brought forward by different epistemologists. We hope shortly to publish the results of an inquiry in "Modern Theories of Knowledge," where this question has been fully discussed; here we can only bring forward a few names out of a host that might be quoted, and we cannot attempt *to prove* that their views are related.

W. Schuppe deserves the first mention. According to him, the content of consciousness may be non-psychical and trans-subjective, that is to say, it need not fall within the *individual* psychical life.[1]

In *Rehmke* we have an equally important representative of a similar theory of consciousness. Here too belongs *Avenarius*' doctrine of "co-ordination of principles" (*Prinzipialkoordination*) by which he means the co-ordination of the "central organ" (*Zentralglied*) or Ego and of its opposite (*Gegenglied*) as members of a whole, a doctrine which he has developed in his work: *Der menschliche Weltbegriff.*

We find a similar theory under quite a different form in the writings of *Windelband* and *Rickert* in so far, at any rate, that Rickert maintains in his theory of supra-individual consciousness the non-psychical and non-subjective character of some contents of consciousness. Within the Marburg school of transcendental Idealism, *Natorp* is developing a new theory of consciousness. The new concept of consciousness is also emphasized in the works of the present writer, more especially in *Grundlegung des Intuitivismus*, and *Grundlehren der Psychologie vom Standpunkte des Voluntarismus.*[2]

In modern Psychology also the new theory of consciousness is frequently to be met with. We find it under different forms in the works of *T. Lipps*; in the writings of *Pfänder*, a disciple of Lipps, especially in his *Einführung in die Psychologie*; in a very remarkable treatise by *Stumpf*, entitled *Erscheinungen und psychische Funktionen*; and also in the inquiries carried on by the school of Psychology founded by *Külpe.*

[1] See Schuppe: "Erkenntnistheoretische Logik," 1878; "Grundriss der Erkenntnistheorie und Logik," 1894; and "Begriff und Grenzen der Psychologie," in *Zeitschrift für imm. Phil.* 1896.

[2] These appeared for the first time in Woprossy *Philosophii i Psychologii*, in 1902-1903 and 1904-1905, German trans. *Die Grundleg. d. Int.* by Max Niemeyer, Halle, 1908, and *Die Grundlegung d. Psychol. v. St. d. Volunt.* Barth, Leipzig, 1904.

In the present treatise we shall discuss the bearing on Logic of this reform of the concept of consciousness, dealing with it from the standpoint taken up in our previous works, more especially in *Grundlegung des Intuitivismus.* We must therefore begin by making quite clear what it is which distinguishes this conception of consciousness from others.

Consciousness, as already explained, is the sum-total of all contents to which the Ego stands in a certain unique relation, which may be metaphorically indicated by the verb "to have." Of this "having," however, there are two kinds, clearly distinguishable from one another: in some cases the conscious content is clearly an *expression of my Ego* (as, for example, joy, wish, etc.); but in others it seems to *confront the Ego as something alien to itself*, and only falls within the field of my consciousness as something "given to me" in so far as I direct my attention to it, and to the extent that I keep it in view (as "red," "hard," and so on). We may call the first kind of content "mine," and the second the "given-to-me." [1]

But "my" states of consciousness (joys, wishes) may, in their turn, become the object of my attention, the object which I have before me. In this case they stand in a two-fold relation to my Ego, and are connected with it in two different ways, being both "mine" (in the sense that they are "my states") and "given-to-me" as an object for consideration.

The second kind of "having-in-consciousness" is the most important for the Theory of Knowledge.

In this kind of consciousness there exists a relation between the Ego and some content or other; it makes no difference whether this content be a psychical or material phenomenon, whether it fall within the sphere of the subjective (the sphere of the psychical life of an individual) or of the trans-subjective world. We will call this unique relation between the Ego and something which owes its existence to my attention, and leads to that something being "given-to-me": *perception*, *intuition*, or *gnoseological co-ordination* between subject and object.

[1] This distinction of the content of consciousness into two classes may be met with under various forms in Schuppe's Theory of Knowledge, in the Psychology of the school of Lipps, and in my works: *Die Grundlehren der Psychologie vom Standpunkt des Voluntarismus* and *Die Grundlegung des Intuitivismus.* It has been worked out quite recently under a similar form by *D. Michaltschew*: *Philosophische Studien*, 1909, s. z.B. pp. 31-39.

This relation is *not a causal relation*; it does not consist in an *effect* of the non-Ego on the Ego, or conversely of the Ego on the non-Ego. For this reason the scientist who is accustomed to deal exclusively with the relations which have been established by Physics, Chemistry, and Physiology, and are recognized in the mechanical view of the world, and who is unfamiliar with the kind of relations which underlie all cognitive processes, finds great difficulty in accepting the theory of the structure of consciousness here propounded.

Now since the fact that a content is "given-to-me" does not imply any causal interaction between subject and object, there are no longer any grounds for maintaining that every perceived content must be sensuous in its nature, that is to say, must be made up of sensations. According to this theory, space, time, and motion are regarded as *non-sensuous*; nevertheless, although they cannot in any sense be described as elements of knowledge derived from sensations, they are "given-to" perception. Further, this theory leads us back at several points to the Platonic doctrine of the contemplation of ideas.[1]

But in being explained as a relation of "being-given-to" between the subject and the known object, the true nature of consciousness is not restricted to the three already mentioned; namely, the Ego, the content of consciousness and the relation between them (perception or epistemological co-ordination). As already pointed out, the Ego brings the activity of attention into play; in other words, it brings about a subjective psychical state in an individual. Further, in order to become aware of the "given" content, the subject knowing must perform an act; he must set the "given" content over against other contents in order to compare it with them. From this point onwards knowledge of the object is gained by means of this process of comparing and distinguishing; but such knowledge must always be *incomplete*, for it only reveals to us one side of the object, namely, the one apprehended by means of comparison (colour, form, and so on).

Every time an object is thus apprehended a definite judgment concerning it is formed, and if we call the matter of this judgment the *cognitive content* we may sum up the results of the preceding analysis as follows: In every process of

[1] This result of Intuitivism is developed in my treatise: Die Unsterblichkeit der Seele als Problem der Erkenntnistheorie; *Woprossy Philosophii*, vol. 104.

judging three elements can be distinguished in addition to the subject knowing; these are the object, the content, and the act of cognition. When, for example, I say, on perceiving the swing of a pendulum, "the pendulum is swinging," the object of knowledge here is that infinitely complex segment of reality which is denoted in this judgment by the term "the pendulum." (This segment of reality is denoted by the word "pendulum" in this judgment in virtue of knowledge which was gained in an earlier judgment, "This is a pendulum.") In this case the content of the judgment is the swinging of the pendulum; the act of knowledge is made up of the attention directed to this object and its motion, and also of the work of comparison.

We must draw a sharp dividing line between the act of knowing on one side and the object and content known on the other: the act of knowledge is always a *psychical* state of the subject knowing and bears the character of an *event* (in other words, it is temporal), which comes to pass at the moment in which the judgment is formed. On the other hand, the object and content of knowledge may be non-psychical, trans-subjective and may belong to a *different point of time from the cognitive act*; indeed they may even, as in the case of *mathematical ideas*, be *timeless*.

These two sides of the process of knowing may be described as follows: The cognitive act is the side of the *psychic individual*, it is the *subjective* side of knowledge, while the object and content of the cognition constitute its *objective* side. In every act of knowing, besides the object and the content of the cognition (the objective side of knowledge) there must also necessarily be the individual's psychical act of knowing (the subjective side). This gives rise to an illusion; the individual knowing believes these different sides of the cognitive process to form one inseparable whole. To the influence of this illusion is due the *erroneous transference of the qualities of the cognitive act to the object and content known*; in other words, the transference of qualities from the subjective to the objective side of knowledge. Thus arises the erroneous conviction that object and content, like the *act* of knowledge, must be *psychical* states of the *subject*, happening *in time*, and even *at the same time* as the act of knowing. And in this is founded the theory that the existence of things beyond the moment in which they are perceived cannot be proved—that it is impossible to prove their *continuous existence*.

Here we have the root of all the different theories which have been framed to show that everything which exists is *immanent* in consciousness. If we remember what has been said above, however, we shall see that the differences between the immanent and the transcendental theories of knowledge are not irreconcilable. We must bear in mind the distinction already insisted upon between the subjective and objective sides of the act of knowing and must further remember that the temporal qualities of the act of knowing do not necessarily belong to the object and content of this act. We can then admit that *the object and content of knowledge may be transcendent as over against man, the knowing subject, or even as over against any consciousness whatever, although in the moment of framing a judgment they become immanent in the consciousness of the subject knowing. This immanence is brought about through the functioning of the individual,* who brings attention to bear on the object and content in question.

If we accept this view of the structure of consciousness we shall have no difficulty in determining the fundamental qualities of truth—identity, eternity and universal validity; in other words, independence of the individual knowing. According to the theory of Intuitivism a true knowledge of a constituent *A* of the world (an idea, an event, and so on) is gained whenever, in virtue of an act of cognition directed towards it, this element becomes the object and content of knowledge; that is to say, when it enters into an act of cognition as its objective side. However many individuals are perceiving, and at whatever times this perception takes place, as long as it has really been perceived and not merely fancied, it is always *the same A* which forms the content of the judgment: in other words, truth is identical, eternal and universally valid. The advantage over other theories, for example that of Husserl, possessed by this theory is that it is not driven to assume a *reduplication of the world*; it need not posit a *separate and ideal realm of truths.* According to our theory even a changing and temporal content, *in so far as it is considered in relation to the act of knowing*, may be a truth; that is to say, it has an *eternal, identical and universally valid meaning.* This result is not obtained by transforming a temporal element of the world into a timeless *idea*, but by admitting a specific and *ideal relation* (observation) between the subject knowing and the object known.

II. THE TRANSFORMATION OF LOGIC.

The theory of the structure of consciousness given above entails important modifications within the sphere of Logic. We can here only look at some of the most important of them :— the doctrine of synthesis and analysis, the principle on which judgments are based, and the fundamental principle of all rational inference.

1. *Analysis and Synthesis.*

In modern philosophical literature the doctrine that knowledge implies synthesis as well as analysis is widely held. Analysis is certainly an activity of the individual knowing. But whence comes synthesis?

If we say that synthesis is also the result of an activity of the individual knowing we obviously commit ourselves to Psychologism and Subjectivism.

But if, with the *Kritik der reinen Vernunft*, we assume that the synthetical character of cognition is organically connected with the unity of consciousness, we are also committed to a psychologism, although a disguised one, concealed under the expression "scientific consciousness" and the like. If we are to make war in earnest on Subjectivism as well as on Psychologism, we must emphasize the distinction between the cognitive act on the one side and the object and content of the cognition on the other; or, in other words, between the process of becoming conscious and the conscious content, between the act of perception and that which is perceived, etc. According to this theory, analysis is the work of the individual knowing, but synthesis is given in the constitution of the object. That process of thought which we call *comparison* is exclusively a *psychical* process; it goes on *in the individual* and terminates in the *analysis* of the object. The object "given-to-me" to analyse, on the other hand, is a highly complex piece of reality, the product of a *synthesis* which must be the work of somebody, though certainly not of myself as an individual. And the aim of comparison is to *disentangle the various sides of this complex object.* In other words, cognition is a work of analysis performed by the individual which leads to *the tracing out of the system of synthesized relations* which constitutes the object.

Regarded from its subjective side, or *according to its origin*

in the psychical life of the individual, cognition is always the result of *analysis*, while the objective side of cognition is always *synthetic* in character.

The science of Logic investigates the grounds on which the relation between the subject and object of a judgment are based, and is only concerned with the objective or synthetical side of cognition; analysis, which, as we have seen, is the subjective work of the individual knowing, and only serves as a bridge to the objective side of judgment, is of importance for Psychology, but not for Theory of Knowledge. As a matter of fact, however, owing to the confusion of the individual-subjective with the objective side of cognition, Logic occupied itself almost entirely, until Kant's time, with the problem of analysis. Hence, owing to a series of misconceptions, there grew up a tendency to place the relation of identity in the forefront, and to let that of ground and consequent recede into the background. In so doing Logic did not forget the relation of ground and consequent, but transformed it (as, for instance, the Rationalists did) into that of identity. Associated with this was the further tendency to place in the foreground the Laws of Identity, of Contradiction and of Excluded Middle, which we may call the *analytico-logical laws of thought*, while the *synthetico-logical law of Sufficient Ground* fell into neglect. Since Kant's time interest in the synthetical side of cognition has been on the increase; nevertheless, even up to the present day, the attitude of logicians on this question has remained indeterminate, especially on such points as mediate inferences and those judgments which do not rank amongst the principles of knowledge. Hence we will pass on here to a somewhat closer examination of judgment and syllogism.

2. *Judgment and Syllogism.*

Most logicians and epistemologists are agreed that an object is cognized by means of a distinguishing and identifying activity which is directed towards it,—that is to say, by means of a comparison with other objects.

This is a correct account of the process from the psychological side, but it becomes incorrect when, misled by the tendency already mentioned, philosophical writers transfer this explanation to the objective side. For the notion then arises

that cognition is the product of a comparison of *subject* and *predicate*, or of premisses and conclusion, so that the meaning and logical ground of a judgment consist in the establishing of a relation of *identity* (generally only partial) or of *contradiction* between subject and predicate. According to this theory a judgment—*e.g.* "the rose is red," is constructed as follows: the subject is the *perception* of the red rose, and the predicate is added to it by means of the partial identification of the *idea* of "red" with the perception: "red rose." Similarly it may be said that the syllogism: "Natrium is a metal, metals are conductors of electricity, therefore natrium is a conductor of electricity" is based on the principle of identity, for it asserts a partial identity between either the denotation or the connotation of the concepts "natrium," "metal" and "conductor of electricity."

Jevons' *Principles of Science* affords a classical example of this tendency in Logic. It is worthy of remark that even avowed disciples of Kant, and logicians who have been strongly influenced by him, although they recognize the categorical synthesis as the basis of the objective side of knowledge, yet do not consistently carry out this principle in their logical systems. They often turn back, in a hesitating sort of way, to the principle of Identity as the logical ground of the judgment, smuggling it in under such terms as the "agreement" of the predicate with the subject, the "logical immanence" of the predicate in the subject, and so on.

But we must now give a concise and clear formulation of the most important characteristics of these systems of Logic. They imply a return, conscious or unconscious on the part of their author, to the doctrine of the Rationalists, who only allow a judgment to be logically grounded when *the combination of subject and predicate is determined by an analytical necessity.* A typical form of this kind of necessity occurs in the analytical judgment "*SP* is *P*" (if indeed such statements deserve to be called judgments). An essential property of this judgment and of *all judgments which are analytically necessary* is that an appeal to the three logical laws of thought—the axioms of Identity, of Contradiction and of Excluded Middle—suffices to establish the necessity and universal validity of the judgments based upon them.[1]

[1] It was on this account that we agreed to call them the *analytico*-logical Laws of Thought.

Modern logicians are well aware that not every judgment can be transformed into an analytical one. But by means of a curious *theory of inference* they have discovered a way of exhibiting, at any rate, the *deduction* of synthetical judgments from synthetical premisses as the result of an analytical necessity. Of course, where universal synthetical premisses are taken as given, without any formal logical grounding—especially premisses which underlie all knowledge, such as the Law of Causation, the axioms of Geometry, etc.,—there is no difficulty in showing that between such universal synthetical premisses and the synthetical conclusions drawn from them which are less general in character, there exists a relation of analytical necessity—(a secondary law, for instance, can be subsumed under a more general law).

In the great majority of modern systems of Logic this is precisely how the relation between premisses and conclusion is explained. This is evident from the fact that they nearly all contain statements such as the following: The conclusion must not contain more than do the premisses (no addition is logically justifiable), or, no term must appear in the conclusion which was not contained in the premisses (since there would be no logical ground for its appearance), and so on. The conviction underlying all such statements is that a conclusion is only logically established when it follows from the premisses by an *analytical* necessity. But if this is so it amounts to saying that whenever premisses are accepted as given it is only necessary to appeal to the Laws of Identity, of Contradiction and of Excluded Middle to compel every thinking being to admit the conclusions drawn from them. In a logical theory so constructed it is clear that there is no need for the fourth law, the Law of Sufficient Ground (Reason). But since it has become customary to quote it, many logicians, although it is not necessary to their systems, cannot make up their minds to discard it. And so we get the following situation: out of deference to custom logicians bring this axiom into their systems, but when there they do not know what to do with it. And if they do find an application for it it is only by reading into it the contents of the Laws of Identity, Contradiction and Excluded Middle; by exhibiting it, that is to say, as a law which demands that only such judgments as are determined by an analytical necessity shall be recognized as

adequately established. It is taken merely as an epitome of the other three laws. That with such an interpretation the Law of Sufficient Ground has retained even an illusion of independence is perhaps due to the fact that it is generally applied negatively, to prove the invalidity of such inferences, for example, as: "A man is a human being, a woman is not a man, therefore a woman is not a human being."

In proof of what we have been saying, we may quote the "Logik" of Alexander Wedenskij and his treatise: *Ein neuer und leichter Beweis des philosophischen Kritizismus* (*Archiv für syst. Philos.* 1910). These examples are especially significant because they are taken from a system of Logic which has been built up with great consistency on a Kantian foundation. They show us clearly how, with regard to the problem we are now examining, the modern Kantians remain true to their master in insisting on the synthetical character of objective knowledge when they are dealing with the principle of knowledge, but abandon it when they come to the logical laws of thought and the process of inference.

Wedenskij's theory of inference is based on his doctrine of *logical coherence.* "The connexion between the conclusion and the premisses," says Wedenskij, "or the dependence of the conclusion upon the premisses in valid syllogisms is generally called logical coherence or logical dependence." [1]

But in what does this logical coherence consist? By way of an analysis of syllogism Wedenskij comes to the conclusion that it means nothing more than the coherence demanded by the Laws of Contradiction and of Excluded Middle. This coherence exists, Wedenskij says, not only between premisses and conclusion; it may also exist between concepts: "thus it exists between the concept of a square and the concept of rectangularity; for were we to refuse assent to the judgment that a square is a rectangular figure, we should be obliged to conceive a square as not rectangular. Such a judgment, however, is in conflict with the content of the concept of a square, and, since a contradiction is unthinkable, cannot be thought. Hence we see that that is logically necessary which cannot be denied without leading our thought into contradiction with itself (the occurrence of such contradiction is, according to the Law of Excluded Middle, inevitable)." [2]

[1] *Logik,* § 6, p. 6 (Russian edition), 1910.

[2] *Op. cit.* § 134, p. 95.

It is clear from this passage that by logical coherence and logical necessity Wedenskij means what we have called analytical necessity. Let us now see how Wedenskij proves that precisely the interconnexion required by his theory obtains between the premisses and conclusion, and how he establishes the necessity of the axiom of Sufficient Ground to explain the rules of Inference. "The conclusion of a legitimate inference," he says, "is only necessarily valid (in virtue of our agreement with what preceded it) in so far as we admit that that which is in contradiction with itself cannot be thought. For instance, if we have admitted that quicksilver is a fluid and that all fluids are elastic, a natural compulsion obliges us to admit that quicksilver is elastic. But why is this necessary? Because we hold a self-contradiction to be unthinkable; if we reject the elasticity of quicksilver we either deny that it is a fluid or that all fluids are elastic, and yet we have already admitted both these judgments. But if, on the other hand, we maintain that self-contradictions are thinkable there is nothing to prevent us from concluding as follows: Although quicksilver is a fluid, and all fluids are elastic, yet quicksilver is not elastic. If we assume that fluids are not governed by the Law of Contradiction, or that we do not know whether they are governed by it or not, there is no necessity to admit that quicksilver is elastic, even though we have admitted that it is a fluid, and that all fluids are elastic.

"I must not, of course, be understood here to mean that the necessity of admitting the conclusion of a correctly-drawn inference is *only* conditioned by the Law of Contradiction. This necessity also follows from the Law of Excluded Middle. If we could neither affirm nor deny the predicate 'elastic' of the subject 'quicksilver,' but could instead relate the predicate to the subject in a way which allowed of simultaneous affirmation and negation (let us agree to symbolize this third relation by the copula X instead of 'is' and 'is not') we should then be at liberty, while accepting the premisses, to reject the conclusion, and could do this without contradicting the premisses. In such a case we could say, 'although quicksilver is a fluid, and all fluids are elastic, quicksilver is neither elastic nor not-elastic, but X-elastic.'

"Still it is evident that logical necessity is only immanent in valid inferences in virtue of the Law of Contradiction.

Hence I say that valid inferences are necessarily true only in so far as we hold self-contradiction to be unthinkable."[1]

Thus the conclusion is based on the Laws of Contradiction and of Excluded Middle. Nevertheless, these laws do not suffice. "If the Law of Sufficient Ground is excluded from Logic, difficulties arise; in the absence of this law the rules of the syllogism cannot be justified. Only by its help can it be shown, for example, why it is incorrect, that is to say, logically illicit, to have a negative minor premiss in the first figure,—why, for instance, it is not permissible to argue: Man is a human being, woman is not a man, therefore woman is not a human being. In this case the Laws of Identity, Contradiction and Excluded Middle are in no way violated, and yet the conclusion is invalid. Why is this? Simply because the Law of Sufficient Ground is violated; because the given premisses contain no ground sufficient to compel our consent to the conclusion."[2]

"But in cases of valid syllogism, the premisses contain a ground which is sufficient, in virtue of the Laws of Contradiction and of Excluded Middle, to compel us to assent to the conclusion which follows from them."[3]

Further on, Wedenskij formulates the law as follows: "Thought accepts as true only those judgments for which there exists a Sufficient Ground to compel our acceptance."[4]

From all the preceding quotations, and more especially from the last but one, we see that according to this interpretation the axiom of Sufficient Ground has no independent and positive content; it is a mere summation of the three other laws and is used more especially to justify *prohibitions*;—for example, of a *quaternio terminorum* or of concluding from two affirmative premisses in the second figure of the syllogism.

But this is as if a physicist were to enunciate Boyle's Law: "The temperature remaining the same, the pressure of a mass of gas varies inversely as the volume," and were to add: "and there is another law which states that when the temperature varies the pressure of a mass of gas is not inversely proportional to its volume."

It remains to be pointed out that this interpretation of the

[1] A. Wedenskij, "Ein neuer und leichter Beweis des philosoph. Kritizismus," *Archiv. für system. Phil.* 1910, p. 203.

[2] *Op. cit.* p. 197. [3] *Logic*, p. 96. [4] *Op. cit.* p. 96.

theory of syllogism can only dispense with the Law of Identity in so far as author and reader, while apparently occupied with the Law of Contradiction and the negative side of the process of inference, involuntarily read into this discussion the more significant positive side which is expressed by the Law of Identity.

The assertion that the conclusion "not-*C*" is an illicit inference from the premisses *A* and *B* because it is in contradiction to the thought contained in the premisses is really equivalent to the assertion: The premisses *A* and *B* contain as part of their content the thought *C*; hence the conclusion *C* is obligatory on every thinking being *in virtue of the partial identity between it and the premisses.*

Moreover, Wedenskij could not have dispensed with a direct appeal to the axiom of Identity but for the fact that he followed a by-path: instead of comparing the correct conclusion "quicksilver is elastic" with the premisses he took the *erroneous* conclusion, "quicksilver is not elastic." He put this by the side of the premiss, "quicksilver is a fluid," and then proceeded to draw, under the third figure of the syllogism, the *new conclusion*, "some fluids are not elastic"; or, according to the method of so-called "immediate inference" the conclusion, "quicksilver is no elastic fluid." He then compared these conclusions with the premisses of the syllogism and found that they contradicted one or other of them. But the *reductio ad absurdum* is not permissible here. If Wedenskij's opponent had challenged the necessity of the first conclusion he would also have challenged that of the second, and could never have been led into self-contradiction by this means.

The logical systems which reduce all the logical grounds of judgments to an analytical necessity convey the impression of being strictly scientific. Their exhibition of the logical moment contained in the ground of all judgments appears ideally convincing and irrefragable. Nevertheless, it is not difficult to show that this cogency and exactitude are only apparent. As a matter of fact, the identity and absence of contradiction which compel assent to the judgment, "*S* is *P*," can *only be proved by means of a judgment previously established*; but since this process cannot be continued *ad infinitum*, some judgment, concept or idea must have served as a starting-point which was itself incorporated into the system of knowledge without being grounded in an analytical necessity.

Hence those thinkers who maintain that a judgment is only logically grounded when it is based on the analytico-logical Laws of Thought are compelled to regard the starting-point of every system of knowledge as logically *un*grounded. This admission is, indeed, generally made in the logical systems under consideration ; they give as examples of such logically ungrounded starting-points, innate ideas, synthetic judgments *a priori* or judgments of perception. In these systems the theory of certain knowledge is either necessarily dualistic or else, treating the starting-point of knowledge as not certain, they are obliged to present every system as merely hypothetical: "If A is true, then B, C, D, which follow analytically from A, are also true."

Let us now pass on to examine the structure of the *subjects* from which a new truth is won by means of an analytical necessity. It would be senseless to affirm a judgment in which the predicate was entirely identical with the subject, or to draw a conclusion completely identical with one or both of the premisses. Hence the logicians who regard analytical necessity as the only logical ground always have in mind the cases in which a *partial* identity exists between premisses and conclusion. The starting-point is therefore always more complex than the concept or judgment which is to be deduced from it. It contains the element P, which is an essential part of the inference about to be drawn, and, in addition, a certain supplementary S; it is, that is to say, a synthetical whole, SP. The analysis which lays bare an analytical necessity shows us that it implies at its starting-point an already existing synthesis.[1]

It must be remembered that the synthesis between S and P must always be a *necessary* one ; in other words, S must be an element of the ideal or real worlds, such that it cannot exist without P, so that it is really true that "when S is P is also." As a matter of fact, the analytical necessity "when SP is P is also" has only a value for knowledge in the cases in which the necessary synthesis is given in the subject (or in the premisses), and this is the *synthetical necessity of combination*, which can be expressed by the formula "when S is P must also be." In other cases, when S need not necessarily be combined with P

[1] For valuable remarks on this point, see Natorp, *Die logischen Grundlagen der exakten Wissenschaften*, pp. 7-11.

(as, for example, the concepts "rose" and "red"), the judgment in question would be absolutely meaningless: "those *S*'s which are *P* are *P*" ("those roses which are red are red"). But we have said enough. The emptiness of the conclusions drawn under similar circumstances is still more apparent; but our space does not admit of exemplification. To sum up what we have been saying: *Every analytical necessity which is of value for knowledge is necessarily based on a synthetical necessity.* It cannot be otherwise: an analysis is only possible when a synthesis is already given.

The presence of a necessary synthesis between *S* and *P* can be detected by the fact that whenever *S* is explicitly mentioned *P* is always tacitly admitted; it is enough, for example, to think of an "equilateral triangle" instead of an "equiangular, equilateral triangle." Hence a judgment (or a syllogism) that itself contains an analytical necessity, and whose starting-point contains a necessary synthesis, cannot be thrown into the patently analytical form, "all *SP* is *P*," but must appear under the form which at any rate has the appearance of being synthetic, "*S* is *P*." Moreover *in practice* we shall always find that the analytical formula expresses a *theory* invented by Logic, while in real life judgments and syllogisms are always expressed in the *synthetical* form, "*S* is *P*," or in the form of a series of premisses of which the conclusion has not yet been explicitly drawn.

But if this is true a doubt creeps irresistibly into the mind: may it not be that judgments and syllogisms are based not only apparently, according to their form, but also really on a synthetical necessity? Is analytical inference merely a logical invention? In practice, the synthetical necessity may easily be confused with the analytical. If the content of the concept *S* is necessarily combined with that of the concept *P*, the synthetical judgment "*S* is *P*" is in danger of being regarded as analytical, for it may be said that *P* is implicitly thought in *S*, or, in other words, that *S* is changed into *SP*. Similarly, if I think the premisses "*S* is *M*, and *M* is *P*," I cannot help drawing the conclusion "*S* is *P*"; hence it is easy enough to transform the synthetical necessity (which consists in this, that the consequent of the consequent is the consequent of the ground) into an analytical one, *by adding the conclusion to the premisses in thought*, and comparing the conclusion, not with the premisses

"*S-M*, *M-P*," but with premisses in which the conclusion was already incorporated. In this way, even identity is preserved.

We therefore maintain that the reduction of judgments and syllogisms to an analytical necessity is only possible as the result of a previous artificial transformation of a synthetical into an analytical judgment ; we further maintain that such a transformation serves no purpose and explains nothing. For indeed all truths that are based on an analytical necessity have always the *same* truth as their basis, and this truth is grounded in a synthetical necessity. In other words, analytical truth is only valuable when it is a *correctly performed analysis* of a *correct synthesis*. It is only in the case in which the starting-point of my knowledge contains (in the subject of a judgment or the premisses of a syllogism) a synthesis *SP*, which has objective validity, that the comparison between it and *P* yields an analytical necessity which is objectively valid. A comparison of *P* and *SP*, based on the Law of Identity, is an easy task which can be faultlessly performed. The difficulty consists, not in this process, but in choosing a true synthesis as a starting-point. Hence a logical system which evades the problem of synthesis, which asserts that, the premisses being given (from whence is a matter of indifference), its task is restricted to the examination of the ground of the judgments deduced therefrom, such a system, I say, merits the reproach that the task it has undertaken is not merely easy but superfluous—for it only consists in showing *how already known truth can be repeated correctly.*

As a matter of fact, if the sole means of logically establishing a judgment is analytical necessity, and the process, as we have shown above, only consists in repeating something already known, it may be caricatured as follows: If I know, *by experience*, that water is composed of oxygen and hydrogen, this knowledge rests on no logical foundation; but if I say (in more or less indirect forms of speech) "Water is composed of oxygen and hydrogen, hence oxygen and hydrogen compose water," my knowledge becomes a *logically grounded* scientific fact. No doubt for this sort of grounding an appeal to the Laws of Identity, Contradiction and Excluded Third suffices, and there is no part left for the Law of Sufficient Ground to play.

As far as its "logical" side is concerned such a grounding,

it is true, appears ideally correct and certain; practically, however, it is useless. So long as Logic continues to propound these theories, criticism such as that which was directed against the traditional theory of syllogism by J. S. Mill will remain in force. It does not, of course, follow from this that the *technical* rules laid down by Logic are useless. Such syllogisms, for instance, as "S is M and M is P, therefore S is P" are perfectly correct and useful, although the theories which have been brought forward to explain the principle on which they are based are for the most part worthless.

Our criticism extends to those theories of syllogism which, although they may be classed as theories of "substitution," are yet conceived (though in a lesser degree, as, for instance, by Jevons), as applying the Law of Identity. As a matter of fact, in *e.g.* the syllogism "S is M, M is P, therefore S is P," P, not M, is substituted for M. If this is a *de facto* substitution (and not an inferring of P from S in virtue of the axiom that the consequent of the consequent is the consequent of the ground) it presupposes that M and P are identical. Hence these theories are compelled to base the judgment (the premiss "M is P") as well as the syllogism on the principle of Identity, and are therefore open to the criticism we have brought forward above.

Should any reader have troubled to note the various attempts that have been made to base the certainty of knowledge on analytical necessity, he will be aware how widely this tendency prevails—in spite of frequent dissertations on the synthetic nature of knowledge—in modern logical and epistemological literature. And such a careful observer will have remarked the incompleteness and ambiguity which characterizes the systems embodying this theory. Sigwart's "Logic" may serve, in this respect, as material for criticism. If any of my readers are under the impression that analytical necessity is no longer regarded as the only logical ground, they may convince themselves from that book of the longevity of the prejudice in its favour, and of its capacity for reproducing itself under new forms.

In order to free ourselves from any leaning in that direction it may be well to examine the psychological and other conditions which tend to evoke and foster it. And we will take the opportunity thus offered to formulate the doctrine of logical

coherence and the principle of logical proof which is, in our view, the correct one.

The psychological side of knowledge, the act of cognition, consists, as we have already said, in a *comparison.* In this comparison, samenesses and differences are established; that is to say, an *analysis* is performed. In order that this psychological process may be set in motion the presence of a certain something is necessary *with which the content of consciousness can be compared.* Those thinkers who do not keep the subjective and objective sides of cognition apart take it for granted that the logical structure of a judgment consists in the establishment of a relation of identity or of contradiction between the contents of consciousness compared. And if the conditions governing the process of comparison, on the one hand, and, on the other, the structure of the final product of this process (the objective, logical side of the judgment) are not kept apart from one another, it certainly does appear as though the whole of the elements which formed the subjects of the comparison must be reproduced in the logical structure of the judgment or of the syllogism. But, as a matter of fact, comparison (or distinguishing) only plays the part of a psychological scaffolding; it enables me to distinguish *for myself* the differences between the elements which make up the world and the manner of their inter-connexion; in this way I learn to single out the elements which combine to form unities and to trace out the laws of their combination. But that *in which* I thus distinguish unities does not itself enter into the structure of my judgment; it only appears as the background against which the particular world-content in which I am interested is thrown up. Thus, for example, when I answer for myself or others the question as to the colour of "this rose," I distinguish "the rose" (S_1) according to its colour, from other roses round about it, or from its environment (S_2), and I say "this rose is red."

By dint of comparing the rose in question with *other* roses, a new property—"redness" (P)—has been brought under my notice; and, in the process, I have become impressed by the fact that "redness" *belongs to* "this rose" and not to any other part of the environment. Nevertheless, my judgment only contains S and P and their mutual relation; *it does not include those elements of the environment with which the comparison was made.*

By means of the psychological process of comparison, I have detected the connexion between S and P, but in the structure of my judgment neither identification nor distinction has entered: hence its logical signification consists in the establishment not of a relation of identity or of contradiction between subject and predicate, but of a necessary synthetical connexion between them. And, as a matter of fact, a relation of identity or contradiction between the subject S ("this rose," that is to say, this object which has been determined by previous cognitive acts as "flower," "tree," "rose," but whose colour is still undetermined) and the predicate P (red) does not exist. This is obviously a synthetical judgment, and the relation between its subject and predicate can only be exhibited as a relation of partial identity by recourse to the unnatural method of treating it we have already described, namely, by taking as the subject of the judgment, not S, but SP. It is generally said that the *perception*, "this red rose," serves as the subject of the judgment, and that the predicate is added to it by means of the partial identification of the *idea* (*Verstellung*) "red" with the perception "this red rose."

But the logicians who support this theory forget that, if I still require an act of cognition to establish the colour of the rose, I cannot already have had the perception expressed by the words "this red rose"; and that, on the other hand, if I have had such a perception, I do not require a fresh act of cognition to discover "this rose is red": my perception, indeed, is nothing but a psychologically abbreviated expression of the judgment "this rose is red." Expressing this more accurately, and in more general terms, we may say: the supporters of this theory forget that the *difference between perception and judgment is a psychological, not a logical, one.* Hence it is impossible to save the traditional theory of judgment by placing the synthesis of S and P within the subject of the judgment (or within the perception), and then maintaining that a judgment only arises after the comparison of P with the synthesis PS.[1]

Further, the analysis which results in a judgment is not a comparison between subject and predicate, but a *singling out of*

[1] See, on such points as "this red rose," Lipps' *Grundlagen der Logik*, 1st edition, § 44. That the perception differs in no respect from the judgment on its objective side, see N. Losskij: *Die Grundlegung des Intuitivismus*, chap. vi. "Die Erkenntnis als Urteil."

subject and predicate and their inter-relation from the totality of the contents of consciousness. In other words, the act of judging is an analysis, which seeks to lay bare the synthetic necessity of connection between the contents of consciousness "given-to-me." The logical relation between the subject and predicate of a judgment is not one of identity or of contradiction, but of the synthetical necessity of connection. In order to draw special attention to this point, the judgment should be thrown into the form: "where S is, P necessarily is also." The relation thus abstractly conceived must be called a relation of *functional dependence* (this term has been unduly narrowed down to mean a relation of cause and effect), in so far as we are dealing with combinations of elements (ideas, events, and so on) which lie outside the act of cognition; when, however, we are considering how far this same inter-connexion enters into the structure of the judgment or of the syllogism, it must be called a relation of *ground and consequent*. Between the subject and predicate of a judgment, then, the relation of ground and consequent exists, and this relation expresses a synthetical, not an analytical, necessity of connexion.[1]

All judgments, negative as well as affirmative, can be thrown into the given formula. The judgment: "Quantities formed by the addition of unequals to equals are not equal" means: "where unequal quantities are added to equal quantities the result is an inequality." In other words, here too the relation between subject and predicate is a synthetic relation of ground and consequent. Even identical judgments form no exception. The judgment: "The magnitudes A and B are identical with one another" means: "if the magnitudes A and B are given, so is the relation of identity between them." In these judgments the true predicate is the concept of identity, and not the relation which unites subject and predicate. The symbolism of Mathematics tends to foster an erroneous conception of identical judgments: the formula $a = b$ conveys the impression that b is the predicate of the judgment, and that the sign of equality symbolizes the logical relation between it and the subject. But in reality the concept "relation between the quantities a and b" is the subject of such judgments, while the concept "relation of identity" serves as the predicate, and

[1] The doctrine that the judgment contains a synthesis of ground and consequent is admirably set forth in Lipps' *Grundlagen der Logik*, § 82.

the logical relation between these concepts which invests their combination with the character of a judgment is the *synthetical necessity of connection*, or the relation of ground and consequent.

Nor do judgments of perception form an exception to the general rule. But we shall be inclined to overestimate the characteristics which distinguish these from other judgments if we do not bear in mind the following circumstance: The system of synthetical necessities forms an endless chain, consisting of an infinite number of links, *S-M-N-P-R*. If we take *two non-consecutive* links of this chain—*e.g.* *S* and *R*—the relation between them is still a necessary relation of ground and consequent; nevertheless, when we assert this we do so in virtue of a necessity which is to us "inexplicable," because we are not aware of the intermediate links. Most judgments of perception, where the object perceived falls within the sphere of the real, not the ideal, world, bear this character. "This rose" and its "red colour" really stand in a relation of functional dependence to each other; the existence of "this rose here at this moment" and "its redness" are necessarily inter-connected, but of course only as *the links at either end* of a long chain of intermediate relations.

When, therefore, a functional relation of this kind enters into the structure of the judgment, it deserves as much as any other function to be called a relation of ground and consequent. The necessity of such a judgment is no less stringent than that of a judgment concerning mathematical ideas. The difference between them consists in this: we here assert the necessity without assigning the intermediate links. Before we could make this necessity as clear as a mathematical necessity we should have to perform the *infinitely* complicated task of discovering all these links, in other words, of exhibiting an infinite number of other relations.

The connexion between this doctrine of judgments and the theory of consciousness given at the beginning of this treatise is obvious. It is by means of the former that the approximation between Logic and Ontology implied in the latter is made possible. For, as a matter of fact, *one and the same* element of the world (whether the real or ideal world) is *Ontological* when considered independently of the cognitive act on the part of the subject knowing, but passes over into the sphere of *Logic* as soon as it becomes the content of a judgment. Thus there

exists between the elements which make up the world a functional dependence, and it is this very dependence, in so far as it forms the objective side of judgment, which represents a *logical* inter-connexion, an inter-connexion determined by a synthetical necessity of combination. This approximation of Logic to Ontology is not a form of Panlogism, but the *resolution of the opposition between panlogistic Rationalism and empirical Irrationalism.*

A logical system of this kind regards every assertion as logically established in which the objective content of the subject and the objective content of the predicate are connected by a synthetical necessity of combination, or, in other words, according to a relation of ground and consequent. Hence it takes as its general principle of logical proof the *axiom of Sufficient Ground*, understood as a synthetical logical law.

The existence of such a basis may be recognized by the fact that the predicate (or conclusion) only follows the subject (or premisses) in virtue of the objective content of the subject (or of the premisses). There is no co-operation on the part of the individual knowing; he has only passively to comply with what is "demanded" by the content of the subject of the judgment. Directly the individual knowing, instead of following out the objective synthesis, himself effects a synthesis, thinking gives place to fancy and other subjective activities, and the assertion has no longer any logical ground. Amongst other things, it follows from this that thought can only lead to truth and has no place for errors (for the objective content of the subject can "demand" nothing which is not combined with it in a relation of functional dependence). Error is always the outcome of the substitution of some other activity (fancy, etc.) in place of thinking.

Further, a logical system such as we have been discussing is delivered from a dualistic evaluation of the logical grounds on which the different systems of knowledge are based. For, as a matter of fact, from this standpoint *all* judgments—even those which are taken as starting-points by knowledge, as, for instance, an axiom or a judgment of perception—if they are real judgment, are *logically established*; that is to say, they are based on the axiom of Sufficient Ground, understood as the law of synthetical necessity.

When syllogisms are regarded from this point of view there

is no longer any inclination to follow the custom, so characteristic of the traditional Logic, of ignoring all non-syllogistic forms of (mediate) inference, or, at any rate, of throwing them, by force, if necessary, into the form of a syllogism.[1]

If the logical principle of mediate inference, and of judgment, is the principle of Sufficient Ground, *i.e.* a *synthetical* necessity, we need not be astonished to meet such forms as: "$A = B$, $B = C$, $\therefore A = C$" or: "Thales lived earlier than Anaximander, Anaximander lived earlier than Anaximenes, therefore Thales lived earlier than Anaximenes," where *a concept appears in the conclusion which was not contained in the premisses* ("earlier than" in a different sense than in the premisses).

This must happen whenever the process of inference consists in a *real forward movement*, and is not a mere analytical marking time. The different formulas according to which conclusions may be drawn offer a wide and fruitful field for investigation, which, up to the present, has been neglected by logical writers.

All that we have been saying, however, in no way impairs the significance of the axiom of Identity, but it assigns it another place,—a place which is, perhaps, more important than modern Logic is willing to admit. For Ontology must return to the old traditional form of this axiom "*A* is *A*" or "every element of the world is identical with itself." In Logic this axiom must be formulated as follows: "In all acts of judgment the objective content *A* remains one and the same *A*." In the absence of this law not a single element of judgment would be present,—neither subject, predicate nor functional dependence between them.[2]

With regard to the laws of Contradiction and of Excluded Middle we can here only inquire into the part they play in the process of inference. The conviction that in the process of knowledge as a whole theirs is but a subordinate *rôle* has been already exhibited in detail in my *Grundlegung des Intuitivismus*.[3]

[1] An excellent demonstration of the unjustifiableness of reducing all mediate inferences to the syllogistic form is given by M. Karinskij in his *Klassification der Schlüsse*, pp. 63-76 (Russian).

[2] For a discussion of the way in which this law—when the structure of knowledge we have expounded is admitted—explains the universality and eternity of truth, see my *Grundlegung des Intuitivismus*, chap. vii.

[3] *Op. cit.* chap. x. "Die letzten Grundlagen der Erkenntnis," pp. 324-329.

In our opinion, the function of the axioms of Contradiction and of Excluded Middle in the process of inference (at any rate in most cases,—such, for instance, as the first figure of the syllogism and even in the mode Barbara: Quicksilver is a fluid, fluids are elastic, therefore quicksilver is elastic—) is a regulative and not a productive one; for once it has been established by means of a process of inference and according to the axiom of Sufficient Ground that quicksilver is elastic (because the consequent of the consequent is the consequent of the ground) the judgment "quicksilver is not elastic" can *in no case be admitted.* *Before* the inference is drawn the *rôle* of these two axioms is even more modest. It consists in admitting a choice between the two judgments—either "quicksilver is elastic" or "quicksilver is not elastic." Which of these two judgments is to be accepted is decided by a process of inference in accordance with the principle of Sufficient Ground. And even in the absence of this regulative form, and were a crowd of judgments to demand recognition, yet any two premisses that could be chosen subject to the law of Sufficient Ground, would always yield the same conclusion; "quicksilver is elastic." [1]

Writers on Logic have shown considerable reluctance in allowing the principle of Sufficient Ground to come out of its seclusion, but that a tendency in this direction exists at the present day is evidenced by such works as Wundt's *Logic* and by the discussion of the first and second figures of the Syllogism in Sigwart's *Logic.* From this point of view the closed and abstract system given by Lipps in his *Grundlagen der Logik* is especially interesting. A classification of all the inferences based on the principle of Sufficient Ground is given by Fr. Erhardt in his treatise: *Der Satz vom Grunde als Prinzip des Schliessens.* In our *Grundlegung des Intuitivismus* we have attempted to apply this theory to all the leading logical problems.

Much work still remains to be done, and many thinkers must co-operate in the task before we can hope to have a complete transmutation of Logic in the spirit we have attempted to indicate in this treatise.

[1] Of course the axiom of Identity ensures the identity of the conceptual contents of "quicksilver," "fluidity," and "elasticity."

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